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Constructing gauge theory geometries from matrix models

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Abstract: We use the matrix model — gauge theory correspondence of Dijkgraaf and Vafa in order to construct the geometry encoding the exact gaugino condensate superpotential for the $\mathcal{N} = 1$ U(N) gauge theory with adjoint and symmetric or anti-symmetric matter, broken by a tree level superpotential to a product subgroup involving U(N_i) and SO(N_i) or Sp($N_i/2$) factors. The relevant geometry is encoded by a non-hyperelliptic Riemann surface, which we extract from the exact loop equations. We also show that $O(1/N)$ corrections can be extracted from a logarithmic deformation of this surface. The loop equations contain explicitly subleading terms of order $1/N$, which encode information of string theory on an orientifolded local quiver geometry.

Keywords: Topological Field Theories, 1/N Expansion, Supersymmetric Effective Theories, Matrix Models.

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1. Introduction

Holomorphic terms in the effective action of supersymmetric gauge theories give much desired exact information about their non-perturbative dynamics. Based on geometric engineering through string compactifications, Dijkgraaf and Vafa [1, 2, 3] conjectured that the exact holomorphic superpotential of certain effective $\mathcal{N} = 1$ gauge theories[1] is calculated by the planar diagrams of a matrix model. In the simplest case, an $\mathcal{N} = 2$ $\mathrm{U}(N)$ gauge theory is broken by a tree-level superpotential $W(\phi) = \sum_{k=1}^{n} \frac{g_k}{k+1} \mathrm{Tr}\phi^{k+1}$ to an $\mathcal{N} = 1$ theory with gauge group $G = \prod_{i=1}^{r} \mathrm{U}(N_i)$. Here ϕ is an adjoint chiral superfield, and one assumes that all zeros of W are simple and that $\sum_{i=1}^{r} N_i = N$. An effective theory emerges by integrating out the massive chiral fields. Its effective low energy lagrangian can be expressed in terms of the gaugino bilinear fields $S_i = -\frac{1}{32\pi^2} \mathrm{Tr}\, W_{\alpha\, i} W^{\alpha\, i}$ where i indicates the factors of the unbroken gauge group. The claim is that the effective action as a function of these fields is calculated exactly by the leading terms in the $1/N$ expansion of a matrix model whose matrix potential is given by W.

Technically the field theory results for the case mentioned above are not new. They were obtained by geometric engineering in [5] and partly by factorization of the $\mathcal{N} = 2$ Seiberg-Witten curve [5, 6]. This allows for explicit checks [7] and leads to a particular interpretation of the matrix model. The relevance of matrix model structures, such as the planar loop equations, can also be understood more conceptually using supergraph techniques [8] or anomalies [9, 10]. Further developments can be found in [11]–[14].

In this paper, we shall add matter in the symmetric or antisymmetric representation of the $\mathrm{U}(N)$ gauge group and consider an additional term $\mathrm{Tr}\, \bar{Q}\phi Q$ in the tree-level superpotential.[2] This leads to a richer vacuum structure, which has branches where additional $\mathrm{SO}(N_i)$ and $\mathrm{Sp}(N_i/2)$ gauge factors appear. The conjecture of [1] then gives us a rough recipe for how to obtain a dual matrix model. However, supersymmetric vacua are described by a symplectic quotient, which is equivalent with a holomorphic quotient of the space of F-flat

[1] Some of these results carry over to $\mathcal{N} = 0$ orbifolds [4] of $\mathcal{N} = 1$ gauge theories.

[2] Symmetric and antisymmetric representations are interesting because they are building blocks of chiral supersymmetric gauge theories. In particular many models with dynamical supersymmetry breaking are based on antisymmetric representations, see e.g. [15, 16, 17].

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configurations through the action of the complexified gauge group. Thus the gauge-theory analysis of the vacuum structure leads to solutions in the space of complex matrices modulo complexified gauge transformations. On the other hand, hermitean matrix models have hermiticity constraints on their matrix variables. To make the matrix model useful for the gauge theory analysis, an appropriate constraint should be imposed in a way which allows one to recover the correct vacuum structure of the gauge theory. As an example, consider the one-matrix model associated with the $\mathrm{U}(N)$ gauge theory broken by a cubic superpotential $W(\phi)$. Then the field theory has two classical vacua at $W'(\phi) = 0$, which, for a hermitean matrix ϕ, would correspond to a local maximum and a local minimum for its eigenvalues. Perturbation theory around the local maximum would make no sense in the hermitean model. In this case, the relevant information was extracted in [18] and [19], where it was shown that it matches the gauge theory instanton expansion relevant for both vacua. Using the BRST formalism, the residual gauge symmetry $\mathrm{U}(N)/(\mathrm{U}(N_1) \times \mathrm{U}(N_2))$ acting on the vacuum leads to an expansion scheme in terms of a two-matrix model interacting through the ghosts [7]. To avoid convergence problems, one must reinterpret the expansion around the extrema in terms of a hermitean $N_1 \times N_1$ matrix interacting with an *anti-hermitean* $N_2 \times N_2$ matrix [18]. The resulting perturbative expansion then recovers the leading F-term gravitational corrections to the gauge theory, which in the matrix model arise at order $O(1/N^2)$ [18, 19]. A more systematic analysis of such issues requires a holomorphic construction of the relevant matrix model [20]. Namely it leads to a so-called 'holomorphic matrix model', whose relevance was already pointed out in [1]. As shown in [20], the procedure employed in [18] is indeed perfectly justified in the holomorphic matrix model. In the present paper, we shall encounter similar problems even before choosing a vacuum. Accordingly, we must carefully choose a 'real section' through the space of complex matrixes in order to make all integrals converge. Moreover, we shall need a complex regularization of the matrix model in order to avoid introducing spurious constraints on its filling fractions. This is very similar to what happens in the case of holomorphic A_2 models [20], where use of a complex regularization is crucial in order to avoid similar problems.

In section 2 we introduce the gauge theory model and analyze its classical vacuum structure. Using threshold matching, we also derive the leading logarithmic terms in the effective superpotential.

In section 3 we define the corresponding matrix model, whose action can be taken to be the superpotential of the gauge theory. As mentioned above, it will be crucial to choose a suitable replacement for the hermiticity constraint. Contrary to the one-matrix model, we cannot impose hermiticity of the matrix M which corresponds to the adjoint chiral multiplet ϕ. Indeed, this requirement would prevent the matrix model from probing the complete vacuum structure of the dual gauge theory. As explained in [20], this is not a particular property of our model, but can be traced back to similar issues encountered in ADE matrix models. Using a holomorphic definition and a 'complex' regularization of our model, we shall extract *two* exact loop equations (Ward identities) which are cubic and quadratic in the resolvent. A new feature of these relations is that they contain explicitly terms of order $O(1/N)$ and $O(1/N^2)$. Taking the large-N limit of these constraints allows us to extract a non-hyperelliptic Riemann surface which governs the planar solutions of the model, as well

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as the effective superpotential of the dual field theory. The dual string geometry is expected to be given by an orientifold of the non-compact Calabi-Yau background dual to the A_2 quiver theory.[3] We therefore also give the explicit derivation of the full loop equations for the matrix model based on the A_2 quiver, and relate it to the $\mathbb{Z}_2$ orbifold and orientifolds thereof. However, there are subtleties in the orientifolding, in the large-N transition and in the precise definition of the B-model cycles. We therefore prefer to use matrix model techniques in order to extract the Riemann surface, which should already encode all relevant information about the holomorphic terms in the $\mathcal{N}=1$ effective action. This surface is a triple cover of the complex plane, which cannot generally be written as a hyperelliptic curve. Since we have little guidance from string geometry, we must intrinsically understand this surface, which in the geometric engineering approach would be obtained by integrating out two directions of a dual Calabi-Yau geometry [5]. In particular, one must take into account issues of non-compactness and relations between periods, which are important for a proper count of this curve's parameters.

Similar to the case of $\mathrm{SO}(N)$ and $\mathrm{Sp}(N/2)$ groups [21]–[29], the 't Hooft expansion leads to unoriented ribbon graphs and the large-N expansion of the matrix model is spaced by N and not N^2. Thus the explicit $1/N$ terms entering our loop equations have field-theoretic relevance. To properly analyze such effects, one must implement the filling fraction constraints N_i =const of [1] in the finite N model, which will be done in section 4 by introducing chemical potentials and performing a Legendre transform to extract a microcanonical generating function. This gives a direct derivation of a set of special geometry relations, and provides their finite N generalization. It also naturally leads to a finite N version of Whitham-type constraints.

In section 5 we show how $O(1/N)$ contributions can be computed from a modified Riemann surface, which is obtained by performing a logarithmic deformation of the matrix model. We show that the $O(1/N)$ term of the microcanonical generating function can be computed by differentiating the leading ($N=\infty$) contribution with respect to the coupling constant of such a logarithmic deformation.

In section 6, we check agreement between the matrix model and the field theory. For this, we use the matrix model to compute the leading (Veneziano-Yankielowicz) contribution to the effective superpotential for different branches of the moduli space. This is done by performing BRST gauge-fixing in the manner of [7, 18] and integrating out the quadratic terms in the action. The comparison to the $O(N)$ model and the A_2-quiver is summarized in the appendix.

2. A first view of field theory properties

In this section we study $\mathrm{U}(N)$ $\mathcal{N}=1$ supersymmetric gauge theory with matter in the symmetric or in the antisymmetric representation. The starting point is an $\mathcal{N}=2$ $\mathrm{U}(N)$ gauge theory with matter in the symmetric or antisymmetric representation, $(Q^T, \bar{Q}^T) =$

[3]Factorization of the $\mathcal{N}=2$ curve cannot give hints, since it leads to a completely different branch of the moduli space.

$s(Q,\bar{Q})$. We choose a tree-level superpotential

$$\mathcal{W} = \mathrm{tr}\left(W(\phi) + \bar{Q}\phi Q\right), \tag{2.1}$$

which softly breaks the $\mathcal{N}=2$ supersymmetry to $\mathcal{N}=1$. In many of our considerations below it helps to view the theory as coming from a $\mathbb{Z}_2$-orientifold of the A_2 quiver theory. Orientifolds of supersymmetric A_2 quiver gauge theories have been constructed before with the help of Hanany-Witten type brane configurations in type-IIA string theory [30, 31]. These models were further analyzed in [32]–[35]. Whereas an A_2 quiver gauge theory contains two independent unitary gauge groups and matter that transforms under the bifundamental representation, the orientifolded model identifies the two gauge groups. In addition the orientifold modifies how the matter content transforms. In the A_2 model the chiral superfield Q transforms as $U_1 Q U_2^\dagger$ where U_1 and U_2 are the gauge transformations of the two independent gauge groups. In the orientifolded model this changes to $Q \to UQU^T$. Therefore Q transforms in a two-index tensor representation of $\mathrm{U}(N)$ and the two possibilities of symmetric or antisymmetric representation correspond to the two choices of orientifolds.

In the following we describe the vacuum and phase structure of the emerging $\mathcal{N}=1$ theory and calculate in various phases the Veneziano-Yankielowicz potential from threshold matching.

2.1 The classical moduli space

We are going to classify the possible constant solutions of the classical field equations. Throughout the analysis we also assume that $\mathcal{N}=1$ Fayet-Iliopoulos terms are absent. As is well known, the space of solutions of the field equations can be obtained by minimizing the superpotential W and dividing by the complexified gauge group. The field equations are

$$Q\bar{Q} + W'(\phi) = 0\,, \tag{2.2}$$
$$\phi Q + Q\phi^T = 0\,, \tag{2.3}$$
$$\bar{Q}\phi + \phi^T\bar{Q} = 0\,. \tag{2.4}$$

For quiver gauge theories similar field equations arise and the classical moduli space has been analyzed in [36]. We will adapt the methods there to the case at hand.

Note first that from (2.3), (2.4) it follows that

$$\begin{aligned}\phi^n Q &= Q\left(-\phi^T\right)^n\,,\\ \bar{Q}\phi^n &= \left(-\phi^T\right)^n\bar{Q}\,.\end{aligned} \tag{2.5}$$

We multiply (2.2) from the left with $\bar{Q}$, from the right with Q and commute Q through W' with the help of the previous relation. This gives $\bar{Q}Q[\bar{Q}Q + W(-\phi^T)] = 0$. Now we transpose (2.2) and use it to eliminate $\bar{Q}Q$ and we obtain $-W'(\phi^T)[-W'(\phi^T)+W'(-\phi^T)] = 0$. Finally we transpose this last equation and arrive at

$$\left[W'(\phi) - W'(-\phi)\right]W'(\phi) = 0\,. \tag{2.6}$$

Equation (2.6) will be solved for ϕ being diagonal $\phi = \mathrm{diag}(a_1.\mathbf{1}_{N_1}, \ldots, a_n.\mathbf{1}_{N_n})$ and $\sum_{i=1}^n N_i = N$. The entries a_i have to be the roots of one of the two equations

$$W'(x) - W'(-x) = 0\,, \tag{2.7}$$

$$W'(x) = 0\,. \tag{2.8}$$

The vev of ϕ breaks the gauge group according to $\mathrm{U}(N) \to \prod_{i=1}^n \mathrm{U}(N_i)$. The field Q decomposes into Q_{ij} with Q_{ij} transforming as a bifundamental $(N_i, \bar{N}_j)$ under $\mathrm{U}(N_i) \otimes \mathrm{U}(N_j)$ if $i \neq j$ and as symmetric (antisymmetric) if $i = j$. An analogous statement holds for $\bar{Q}$. From (2.3) it follows that

$$(a_i + a_j) Q_{ij} = 0\,, \tag{2.9}$$

and thus $Q_{ij} = 0$ unless $a_i = -a_j$ and $N_i = N_j$. Such pairs of solutions are indeed generated by the roots of (2.7). Another special root of (2.7) is $x = 0$. Let us study now in more detail the different solutions.

2.1.1 Pair of solutions $a_i = -a_j = b \neq 0$ with $W'(b) = W'(-b)$

In the relevant subspace, Q and $\bar{Q}$ have to be of the form

$$Q = \begin{pmatrix} 0 & q \\ sq^T & 0 \end{pmatrix} \qquad \bar{Q} = \begin{pmatrix} 0 & \bar{q} \\ s\bar{q}^T & 0 \end{pmatrix}. \tag{2.10}$$

A gauge transformation acts on Q as

$$Q \to \begin{pmatrix} 0 & U_1 q U_2^T \\ sU_2 q^T U_1^T & 0 \end{pmatrix}. \tag{2.11}$$

Since U_1 and U_2 are independent $GL(N_i, \mathbb{C})$ matrices, we can bring Q into the form

$$Q = \begin{pmatrix} 0 & \mathbf{1} \\ s\mathbf{1} & 0 \end{pmatrix}. \tag{2.12}$$

From (2.2) it follows then that

$$\bar{Q} = -W'(b) \begin{pmatrix} 0 & s\mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix}. \tag{2.13}$$

The gauge transformations that leave these matrices invariant are given by $(U_1)^{-1} = U_2^T$ and therefore the residual gauge group is $\mathrm{U}(N_i)$ in this branch.

2.1.2 Solutions with $a_i = 0$

This is the special solution of (2.7). The unbroken gauge group in this branch is $\mathrm{U}(N_i)$ and it acts on Q as

$$Q \to UQU^T\,. \tag{2.14}$$

Let us consider first the case of symmetric Q. Since U is a general linear matrix we can choose U such that Q becomes the N_i-dimensional unit matrix $Q = \mathbf{1}$. It follows then that $\bar{Q} = -W'(0)\mathbf{1}$. The gauge transformations that are left over have to fulfill $UU^T = \mathbf{1}$ and are therefore elements of $\mathrm{SO}(N_i)$.

In the case where Q is antisymmetric similar arguments show that Q can be brought to the form

$$Q = \begin{pmatrix} \varepsilon & 0 & \dots \\ 0 & \varepsilon & \dots \\ \vdots & \vdots & \ddots \end{pmatrix} \qquad \varepsilon = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} . \tag{2.15}$$

$\bar{Q}$ is determined by (2.6) and the unbroken gauge group is $\mathrm{Sp}(N_i/2)$.

2.1.3 Solutions with $W'(a_i) = 0$

In this case the equations of motion imply $Q = \bar{Q} = 0$ and the unbroken gauge group is $\mathrm{U}(N_i)$ in this branch.

2.2 The Veneziano-Yankielowicz potential

After having established the structure of the moduli space of vacua we derive now the leading terms in the low energy effective superpotential. Let us take a generic vacuum with $\phi = \mathrm{diag}(0_{N_0}, a_1\mathbf{1}_{N_1}, \dots, a_n\mathbf{1}_{N_n}, b_1\mathbf{1}_{\tilde{N}_1}, -b_1\mathbf{1}_{\tilde{N}_1}, \dots, b_k\mathbf{1}_{\tilde{N}_k}, -b_k\mathbf{1}_{\tilde{N}_k})$. The unbroken gauge group is

$$\prod_{i=1}^{n} \mathrm{U}(N_i) \otimes \prod_{j=1}^{k} \mathrm{U}(\tilde{N}_j) \otimes \begin{cases} \mathrm{SO}(N_0) & \text{for } s = 1 \\ \mathrm{Sp}(\frac{N_0}{2}) & \text{for } s = -1 \,. \end{cases} \tag{2.16}$$

At low energies the non-abelian factors in the unbroken gauge group confine and leave a $\mathrm{U}(1)^{n+k}$ abelian gauge group unbroken. The effective dynamics of the gaugino condensates in the non-abelian factor groups is captured by the usual Veneziano-Yankielowicz superpotential. For the unitary group factors this is

$$W^j_{VY} = S_j \log\left(\frac{\Lambda^{3N_j}_{\mathrm{low,j}}}{S_j^{N_j}}\right) , \tag{2.17}$$

and for the orthogonal or symplectic factor group it is given by

$$W^s_{VY} = \frac{S_0}{2} \log\left(\frac{\Lambda^{3(N_0-2s)}_{\mathrm{low,0}}}{S_0^{N_0-2s}}\right) . \tag{2.18}$$

Taking all gaugino condensates into account, we have

$$W_{\mathrm{eff}} = \sum_{j=1}^{n+k} W^j_{VY} + W^s_{VY} , \tag{2.19}$$

where we substitute $N_j = \tilde{N}_{j-n}$ if $j > n$.

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The low energy scale can be determined by threshold matching at the scales of the masses of the various massive W-bosons and matter fields. In particular we find for the vacuum with gauge group $\mathrm{U}(N_i)$

$$\begin{aligned}\Lambda_{\text{low,i}}^{3N_i} &= \big(W''(a_i)\big)^{N_i} \prod_{j\neq i}^{n} (a_i-a_j)^{-2N_j} \prod_{l=1}^{k} \left(a_i^2-b_l^2\right)^{-2\tilde{N}_l} (a_i)^{-2N_0} \times \\ &\quad \times \prod_{r\neq i}^{n} (a_i+a_r)^{N_r} \prod_{t=1}^{k} \left(a_i^2-b_t^2\right)^{\tilde{N}_t} (a_i)^{N_0} (a_i)^{N_i+2s} \Lambda_{\text{high}}^{N-2s} \,. \end{aligned} \tag{2.20}$$

The first term on the rhs of this equation comes from the fluctuations of ϕ in the N_i'th diagonal block around the vev a_i. The next terms stem from the massive off-diagonal W bosons and in the second line we collected the contributions from the matter fields Q and $\bar{Q}$. Analogously one can analyze the other scale matching relations. We find:

$$\begin{aligned}\Lambda_{\text{low,i+n}}^{3\tilde{N}_i} &= \big(W''(b_i)+W''(-b_i)\big)^{\tilde{N}_i} \prod_{j=1}^{n} \left(b_i^2-a_j^2\right)^{-2N_j} \prod_{l\neq i}^{k} \left(b_i^2-b_l^2\right)^{-4\tilde{N}_l} (2b_i)^{-4\tilde{N}_i} (b_i)^{-4N_0} \times \\ &\quad \times \prod_{r=1}^{n} \left(a_r^2-b_i^2\right)^{N_r} \prod_{t\neq i}^{k} (b_i^2-b_t^2)^{2\tilde{N}_t} (b_i)^{2(\tilde{N}_i+2s)} (b_i)^{2N_0} \Lambda_{\text{high}}^{2(N-2s)} \,. \end{aligned} \tag{2.21}$$

and

$$\Lambda_{\text{low,0}}^{\frac{3}{2}(N_0-2s)} = \big(W''(0)\big)^{\frac{N_0}{2}-s} \prod_{j=1}^{n} (a_j)^{-2N_j} \prod_{l=1}^{k} (b_l)^{-4\tilde{N}_l} \prod_{r=1}^{n} (a_r)^{N_r} \prod_{t=1}^{k} (b_t)^{\tilde{N}_t} \Lambda_{\text{high}}^{N-2s} \,. \tag{2.22}$$

In (2.21) the factor of 2 in the exponent of Λ_{high} reflects the diagonal embedding of the $\mathrm{U}(\tilde{N}_i)$ gauge groups.[4] The higgsing by the vevs of Q which breaks $\mathrm{U}(\tilde{N}_j) \otimes \mathrm{U}(\tilde{N}_j)$ to this diagonal $\mathrm{U}(\tilde{N}_j)$ and the $\mathrm{U}(N_0)$ factor group to either orthogonal or symplectic groups produces also some massive fields. These come however always in the multiplicities of $N=4$ multiplets and therefore do not contribute to the threshold matching. To be specific, consider the breaking to $\mathrm{SO}(N_0)$. The massive fields include W bosons and components of ϕ that lie in the coset $\mathrm{U}(N_0)/\mathrm{SO}(N_0)$. They transform under the symmetric representation of $\mathrm{SO}(N_0)$. In addition Q and $\bar{Q}$ are also symmetric in this case. Thus we count (in $\mathcal{N}=1$ language) $\frac{N_0}{2}(N_0+1)$ vector multiplets and $3\frac{N_0}{2}(N_0+1)$ chiral multiplets that receive masses at the scale set by the vev of Q.

With this method we can of course obtain only an approximation to the exact low energy superpotential. But is it is expected to capture the correct logarithmic behavior at $S_i=0$. In section 6 we will derive this part of the superpotential from a one-loop calculation in a matrix model.

[4]For the threshold matching of the SO/Sp factor group there arises the following well-known problem [37]. Projecting $\mathrm{SU}(N)$ to $\mathrm{SO}(N)$ or $\mathrm{Sp}(\frac{N}{2})$ there are some roots which are invariant under both projections. However, these roots serve as long roots for $\mathrm{SO}(N)$ but as short roots for $\mathrm{Sp}(\frac{N}{2})$. This results in a relative factor of two in the normalization of the roots in the projected groups and this normalization influences the indices of the representations, e.g. for the adjoint representation $C_\theta = (\theta,\theta)g^\vee$, where θ is the highest root and $g^\vee$ the dual Coxeter number.

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3. The matrix model

In this section, we construct the matrix model which is expected to calculate the exact superpotential of the $\mathcal{N}=1$ supersymmetric $\mathrm{U}(N)$ gauge theory with matter in the adjoint and symmetric or antisymmetric representation. Using direct manipulations, we shall derive the exact loop equations of this model. These are two independent identities, which are respectively quadratic and cubic in $\omega(z)$ and $\omega(-z)$, where $\omega(z)$ is the resolvent of the model. The large-N limit of these relations gives algebraic constraints on the planar limit of the averaged resolvent, which lead to a proposal for the algebraic curve governing the dual gauge theory. We also give evidence that the planar vacuum structure of the matrix model agrees with the field theory.

3.1 Construction of the model

Our matrix model results by performing one of two orientifold projections on the matrix model proposed in [2] for the A_2 quiver field theory. The discussion of the covering A_2 quiver theory and that of a related orbifold model can be found in the appendix.

We use the superpotential (2.1) of the $\mathcal{N}=1$ supersymmetric gauge theory with gauge group $\mathrm{U}(N)$, a chiral superfield ϕ in the adjoint representation, a chiral superfield Q in the symmetric or antisymmetric two-tensor representation and a chiral superfield $\bar{Q}$ in the corresponding complex conjugate representation. In order to define the matrix model we have to choose a real section in matrix configuration space. In the one matrix model this is usually done by mapping Φ to a hermitean matrix M (as discussed in [20], this prescription is justified if the relevant superpotential is a polynomial of even degree). Following a similar prescription for our models would lead to a series of problems. Similar to what happens in the quiver matrix models [38]–[41] and the $O(N)$ model [42, 43], we would then have to require that all eigenvalues of M lie on the (strictly) positive real axis. However, we saw in the analysis of the classical moduli space that vacua with negative or zero eigenvalues play an important role. So it seems quite unnatural from the gauge theory point of view to restrict to such matrix configurations.[5] Instead, we shall consider the matrix model:

$$Z_{N,s,\varepsilon} = \frac{1}{\mathrm{Vol}\,\mathrm{U}(N)} \times \int \mathrm{d}M\mathrm{d}Q \exp\left[-N\mathrm{tr}\,V(M) + i\mathrm{tr}\,Q^\dagger M Q\right], \tag{3.1}$$

where the M–integration is performed not over the set of hermitean matrices but rather over the set:

$$\mathcal{M} = \left\{M \in \mathrm{Mat}_N(\mathbb{C}) \middle| M - M^\dagger = 2i\varepsilon\mathbf{1}\right\}, \tag{3.2}$$

where ε is a small positive quantity. We have also imposed the condition $\bar{Q} = -iQ^\dagger$. Together with the shift of the eigenvalues of M into the upper half plane, this renders the Q-integration finite without further restricting the range of M's eigenvalues. Then the Q–integration is performed over the set:

$$\mathcal{Q} = \left\{Q \in \mathrm{Mat}_N(\mathbb{C}) \middle| Q^T = sQ\right\}, \tag{3.3}$$

[5] A similar issue arises in quiver matrix models, as discussed in [20].

where $s=+1$ and $s=-1$ distinguishes the symmetric from the anti-symmetric representation. The measures $\mathrm{d}M$, $\mathrm{d}Q$ are defined through:

$$\begin{aligned} \mathrm{d}M &= \prod_i \mathrm{d}M_{ii} \prod_{i<j} \mathrm{d}\,\mathrm{Re}\, M_{ij}\, \mathrm{d}\,\mathrm{Im}\, M_{ij}\,, \\ \mathrm{d}Q &= \prod_{i<j} \mathrm{d}\,\mathrm{Re}\, Q_{ij}\, \mathrm{d}\,\mathrm{Im}\, Q_{ij} \left(\prod_i \mathrm{d}\,\mathrm{Re}\, Q_{ii}\, \mathrm{d}\,\mathrm{Im}\, Q_{ii}\right)^{\delta_{s,1}}. \end{aligned} \tag{3.4}$$

This model has the gauge-invariance:

$$\begin{aligned} M &\to UMU^\dagger \\ Q &\to UQU^T\,, \end{aligned} \tag{3.5}$$

where U is an arbitrary unitary matrix.[6] Note that we have included the inverse volume of the gauge group into the definition of the partition function (3.1). Also note that we are rather valiant about the convergence of (3.1), which is assured – -with our choice of integration manifold — only provided that W is a polynomial of *even* degree. As in [20], a consistent construction for odd degree potentials would require that we constrain the eigenvalues of M to lie on a certain path in the complex plane, whose asymptotic behavior is determined by the leading coefficient of W. In the present paper, we shall ignore this and related issues, a complete treatment of which requires the full machinery of [20].

Performing the gaussian integral over Q yields

$$\int_{\mathcal{Q}} dQ\, e^{i\,\mathrm{tr}\, Q^\dagger MQ} = \big(i\pi\big)^{N(N+s)/2} \prod_{i<j} \frac{1}{\lambda_i+\lambda_j} \left(\prod_i \frac{1}{\lambda_i}\right)^{\delta_{s,1}} \tag{3.6}$$

where $\lambda_1,\ldots,\lambda_N$ are the eigenvalues of the matrix M.

Next we rewrite the integral over M in (3.1) in terms of eigenvalues of M:

$$Z_{N,\varepsilon,\delta} = C_{N,s}\, \tilde{Z}_{N,s,\varepsilon}\,, \tag{3.7}$$

with

$$\tilde{Z}_{N,s,\varepsilon} = \int_{\mathbb{R}+i\varepsilon} \prod_{k=1}^N d\lambda_k\, e^{-N^2 S} \quad \text{and} \quad C_{N,s} = \frac{2^{-N^2/2+N}}{\mathrm{Vol}(\mathrm{U}(1)^N \times S_N)} \big(i\pi\big)^{N(N+s)/2}\,, \tag{3.8}$$

where the 'effective' action $S(\lambda_1,\ldots,\lambda_N)$ in the eigenvalue representation is:

$$S = \frac{1}{N}\sum_k V(\lambda_k) + \frac{s}{2N^2}\sum_k \ln\lambda_k - \frac{1}{2N^2}\sum_{k\neq l} \ln(\lambda_k-\lambda_l)^2 + \frac{1}{2N^2}\sum_{k,l} \ln(\lambda_k+\lambda_l)\,. \tag{3.9}$$

[6] This can be seen as follows: The action and the measure $\mathrm{d}M$ for the adjoint field are obviously gauge invariant. Therefore we concentrate on the part coming from $\mathrm{d}Q$. It is easiest to switch for a moment to an index notation in which the fields transform as $Q_{ij} \to \frac{1}{2}(U_i{}^k\, U_j{}^l + sU_i{}^l\, U_j{}^k)\, Q_{kl}$, $\bar{Q}^{ij} \to \bar{Q}^{kl}\, \frac{1}{2}(U_k^{\dagger\, i}\, U_l^{\dagger\, j} + sU_l^{\dagger\, i}\, U_k^{\dagger\, j})$. We took care explicitly of the symmetry properties of Q and $\bar{Q}$. The measure $\mathrm{d}Q$ picks up the product of the determinants of the transformation matrices defined in the previous equations. Since the product of the determinants is the determinant of the product we compute the product of the transformation matrices: $\frac{1}{2}(U_i{}^k\, U_j{}^l + sU_i{}^l\, U_j{}^k).\frac{1}{2}(U_k^{\dagger\, m}\, U_l^{\dagger\, n} + sU_l^{\dagger\, m}\, U_k^{\dagger\, n}) = \frac{1}{2}(\delta_i^m\, \delta_j^n + s\delta_i^n\, \delta_j^m)$. This is just the identity in the symmetric and antisymmetric representation respectively and of course the corresponding determinant is one.

In the limit $\varepsilon \to 0$ (which we shall ultimately take below), the factors $e^{-\ln(\lambda_k+\lambda_l)} = \frac{1}{\lambda_k+\lambda_l}$ in (3.8) produce well defined distributions according to the Sokhotsky formula:

$$\frac{1}{\lambda_k + \lambda_l + i0^+} = \mathcal{P}\left(\frac{1}{\lambda_k + \lambda_l}\right) - i\pi\delta(\lambda_k + \lambda_l)\,, \tag{3.10}$$

where we took the limiting eigenvalues to be real in order to make the $i\varepsilon$ prescription explicit and where $\mathcal{P}$ denotes the principal value. Substituting (3.10) into (3.8) (before exponentiating the Vandermonde factors) leads to an expression for the $\varepsilon \to 0^+$ limit of $\tilde{Z}$ as a sum over 'reduced' integrals. This gives a limiting statistical ensemble, which can viewed as the orientifold of the limiting ensemble extracted in [20] for the holomorphic A_2 model.

The potentials $V(x)$ we want to consider are of the form

$$V(x) = t_{-1}\ln(x) + W(x) \qquad \text{where} \quad W(x) = \sum_{k=0}^{d} \frac{t_k}{k+1}\, x^{k+1}\,. \tag{3.11}$$

From (3.9) we see that the effect of integrating only over symmetric or anti-symmetric Q is a logarithmic correction to the potential of order $1/N$. If we introduce

$$\mathrm{U}(x) = V(x) + \frac{s}{2N}\ln(x) \tag{3.12}$$

the action S depends on s only implicitly through $\mathrm{U}(x)$. For the analysis of the matrix model it is useful to further introduce the following quantities. The expectation value of a function $\mathcal{O}(\lambda_1, \ldots, \lambda_N)$ is given by

$$\langle \mathcal{O} \rangle = \frac{1}{\tilde{Z}} \int_{\mathbb{R}+i\varepsilon} \prod_{k=1}^{N} d\lambda_k\, \mathcal{O}(\lambda_1, \ldots, \lambda_N)\; e^{-N^2 S} \tag{3.13}$$

and the eigenvalue density $\rho(\lambda)$ and the resolvent $\omega(z)$ are defined to be

$$\rho(\lambda) = \frac{1}{N}\sum_k \delta(\lambda - \lambda_k) \quad \text{and} \quad \omega(z) = \frac{1}{N}\sum_k \frac{1}{z - \lambda_k}\,. \tag{3.14}$$

The two quantities are related by

$$\rho(\lambda) = \lim_{\nu\to 0} \frac{\omega(\lambda+i\nu) - \omega(\lambda-i\nu)}{2\pi i}\,, \qquad \omega(z) = \int_{\mathbb{R}+i\varepsilon} d\lambda\, \frac{\rho(\lambda)}{z-\lambda}\,. \tag{3.15}$$

3.2 The microcanonical ensemble

The framework of Dijkgraaf-Vafa requires that the model obeys certain filling fraction constraints. In [1], such conditions were imposed only on the large-N microcanonical generating function, which is insufficient in our case since we will have to consider $O(1/N)$ corrections as can be seen from (3.12). This requires that we impose such constraints on the finite-N microcanonical generating function, rather than on its large-N counterpart. The relevant constraints are easiest to formulate by employing a microcanonical ensemble. As we shall see below, the original path integral defines a (grand) canonical ensemble at zero chemical potentials. This allows one to recover the microcanonical generating function by introducing non-vanishing chemical potentials (which are canonically conjugate to the filling fractions) and then performing a Legendre transform to replace the former by the latter.

3.2.1 The (grand) canonical partition function associated with a collection of intervals

Let us cover the displaced real line $\mathbb{R}+i\varepsilon$ with disjoint nonempty line segments Δ_α subject to the condition:

$$\Delta_1 \cup \cdots \cup \Delta_r = \mathbb{R} + i\varepsilon\,. \tag{3.16}$$

We shall let χ_α denote the characteristic function of Δ_α, and consider the filling fractions f_α of Δ_α:

$$f_\alpha = \frac{1}{N}\sum_k \chi_\alpha(\lambda_k)\,. \tag{3.17}$$

This gives the expression of f_α in terms of $\omega(z)$:

$$f_\alpha = \int d\lambda \rho(\lambda)\chi_\alpha(\lambda) = \int_{\Delta_\alpha} d\lambda\rho(\lambda) = \oint_{\gamma_\alpha}\frac{dz}{2\pi i}\omega(z)\,, \tag{3.18}$$

where γ_α is a contour surrounding only Δ_α and not any part of the intervals $\Delta_{\beta\neq\alpha}$. Picking chemical potentials μ_α, we consider the (grand) canonical ensemble associated to our collection of intervals:[7]

$$\mathcal{Z}(t,\mu) = \int_{\mathbb{R}+i\varepsilon}\prod_{k=1}^N d\lambda_k\, e^{-N^2 S_\mu}, \qquad \text{where} \quad S_\mu = S + \sum_{\alpha=1}^r \mu_\alpha f_\alpha\,. \tag{3.19}$$

The original partition function results by setting $\mu_\alpha = 0$. Introducing the (grand) canonical generating function:

$$\mathcal{F}(t,\mu) = -\frac{1}{N^2}\ln \mathcal{Z}(t,\mu)\,, \tag{3.20}$$

we have the standard relation:

$$\frac{\partial}{\partial\mu_\alpha}\mathcal{F} = \langle f_\alpha\rangle = \oint_{\gamma_\alpha}\frac{dz}{2\pi i}\langle\omega(z)\rangle\,. \tag{3.21}$$

In this subsection, the brackets $\langle\dots\rangle$ always denote the expectation value taken in the (grand) canonical ensemble. Since the union of Δ_α covers the whole integration range, the expectation values of the filling fractions fulfill the constraint:

$$\sum_{\alpha=1}^r \langle f_\alpha\rangle = 1\,. \tag{3.22}$$

3.2.2 The microcanonical generating function

Following standard statistical mechanics procedure, we define:

$$S_\alpha := \frac{\partial}{\partial\mu_\alpha}\mathcal{F} \tag{3.23}$$

and perform a Legendre transform to extract the microcanonical generating function:

$$F(t,S) := \sum_{\alpha=1}^r S_\alpha\mu_\alpha(t,S) - \mathcal{F}(t,\mu(t,S))\,. \tag{3.24}$$

[7] In the following μ, S may stand for r-tuples and t for $(t_{-1}, t_0, \dots, t_d)$.

In this relation, μ_α are expressed in terms of t and S by solving equations (3.23). The constraint (3.22) shows that S_α are related through:

$$\sum_{\alpha=1}^{r} S_\alpha = 1\,, \tag{3.25}$$

so we can take $S_1 \ldots S_{r-1}$ to be the independent variables. Then equations (3.23) express μ_α as functions of t and these coordinates, and equation (3.24) implies:

$$\mu_\alpha - \mu_r = \frac{\partial F}{\partial S_\alpha} \quad \text{for } \alpha = 1, \ldots, r-1\,. \tag{3.26}$$

Note that μ_α are only determined up to a common constant shift; this is due to the constraint (3.25) on S_α.

Working with $F(t, S)$ amounts to fixing the expectation values of the filling fractions by imposing the *quantum* constraint (3.23):

$$\langle f_\alpha \rangle = \oint_{\gamma_\alpha} \frac{dz}{2\pi i} \langle \omega(z) \rangle = S_\alpha\,, \tag{3.27}$$

with S_α treated as fixed parameters. This gives a meaning to the procedure of [1] beyond the large-N limit.

3.3 The quadratic and cubic loop equations

The classical equations of motion following from the action (3.9) are $\partial S/\partial \lambda_k = 0$, where:

$$N \frac{\partial S}{\partial \lambda_k} = U'(\lambda_k) - \frac{1}{N} \sum_{l(\neq k)} \frac{2}{\lambda_k - \lambda_l} + \frac{1}{N} \sum_{l} \frac{1}{\lambda_k + \lambda_l}\,. \tag{3.28}$$

The action (3.9) describes a system of charged particles moving along $\mathbb{R} + i\varepsilon$, together with a set of mirror charges of opposite sign. For $s{=}-1$, each particle interacts with all mirror charges excluding its own, while for $s{=}1$ it is also attracted to its own mirror image.

The partial derivatives (3.28) will be used below to obtain a set of Ward identities for the matrix model. In turn, we shall use these identities in order to derive two loop equations for the resolvent $\omega(z)$. Consider the integral:

$$\frac{1}{N^2} \frac{1}{\tilde{Z}} \int_{\mathbb{R}+i\varepsilon} \prod_{i=1}^{N} d\lambda_i \sum_k \frac{\partial}{\partial \lambda_k} \Big(\psi_k(\lambda_1, \ldots, \lambda_N)\, e^{-N^2 S} \Big) = 0\,. \tag{3.29}$$

Differentiating shows that (3.29) is equivalent to the Ward identity

$$\frac{1}{N} \sum_k \Big\langle \Big(U'(\lambda_k) - \frac{1}{N} \sum_{l(\neq k)} \frac{2}{\lambda_k - \lambda_l} + \frac{1}{N} \sum_l \frac{1}{\lambda_k + \lambda_l} \Big) \psi_k - \frac{1}{N} \frac{\partial \psi_k}{\partial \lambda_k} \Big\rangle = 0\,. \tag{3.30}$$

3.3.1 The quadratic loop equation

The quadratic loop equation is obtained by setting $\psi_k = (z-\lambda_k)^{-1}$. Together with the identity

$$2\sum_{k\neq l}\frac{1}{\lambda_k-\lambda_l}\,\frac{1}{z-\lambda_k}+\sum_k\frac{1}{(z-\lambda_k)^2}=\sum_{k,l}\frac{1}{z-\lambda_k}\,\frac{1}{z-\lambda_l} \tag{3.31}$$

one obtains the relation

$$\left\langle \omega(z)^2-\frac{1}{N}\sum_k\frac{U'(\lambda_k)}{z-\lambda_k}-\frac{1}{N^2}\sum_{k,l}\frac{1}{\lambda_k+\lambda_l}\,\frac{1}{z-\lambda_k}\right\rangle=0\,. \tag{3.32}$$

Adding to (3.32) the same equation with z replaced by $-z$ and absorbing the dependence on the potential into $f(\pm z)$ leads to the *quadratic loop equation*:

$$\big\langle \omega(z)^2+\omega(z)\omega(-z)+\omega(-z)^2-U'(z)\omega(z)-U'(-z)\omega(-z)+f(z)+f(-z)\big\rangle=0\,. \tag{3.33}$$

In deriving this we used the identity:

$$\sum_{k,l}\frac{1}{\lambda_k+\lambda_l}\,\frac{1}{z-\lambda_k}+\sum_{k,l}\frac{1}{\lambda_k+\lambda_l}\,\frac{1}{-z-\lambda_k}=-\sum_{k,l}\frac{1}{z-\lambda_k}\,\frac{1}{-z-\lambda_l} \tag{3.34}$$

and introduced the quantity:

$$f(z)=\frac{1}{N}\sum_k\frac{U'(z)-U'(\lambda_k)}{z-\lambda_k}\,. \tag{3.35}$$

In the analysis of the large-N Riemann surface below we will need to know the properties of $f(z)$ in more detail. From the definition of U,V,W in (3.11) and (3.12) it follows that $f(z)$ can be written:

$$f(z)=\bar f(z)-\phi\cdot\frac{t_{-1}+s/(2N)}{z}\,,\qquad\text{where}\quad \phi=\frac{1}{N}\sum_k\lambda_k^{-1}=-\omega(0)\,. \tag{3.36}$$

The function $\bar f(z)$ is a polynomial in z defined through:

$$\bar f(z)=\frac{1}{N}\sum_k\frac{W'(z)-W'(\lambda_k)}{z-\lambda_k}\,. \tag{3.37}$$

Since $\omega(z)\sim 1/z$ as $z\to\infty$, one finds that the large z behavior of $\bar f$ is $\bar f(z)\sim t_d z^{d-1}$. In particular the polynomial $\bar f(z)$ has degree $d-1$.

3.3.2 The cubic loop equation

The cubic relation for $\omega(z)$ results by substituting

$$\begin{aligned}\psi_k&=\frac{1}{N}\sum_m\frac{1}{(\lambda_m+\lambda_k)(z-\lambda_k)}\,,\qquad\text{so that:}\\ \frac{\partial\psi_k}{\partial\lambda_k}&=\frac{1}{N}\sum_m\frac{1}{(\lambda_m+\lambda_k)(z-\lambda_k)}\Big(\frac{1}{z-\lambda_k}-\frac{1}{\lambda_m+\lambda_k}\Big)+\frac{1}{N}\,\frac{1}{4\,\lambda_k^2(z-\lambda_k)}\end{aligned} \tag{3.38}$$

into the Ward identity (3.30). To proceed we make use of the relations:

$$
\begin{aligned}
-\frac{1}{N^3}\sum_{m,k,l(\neq k)}\frac{2}{(\lambda_k-\lambda_l)(\lambda_m+\lambda_k)(z-\lambda_k)} &= \\
&= \omega(z)^2\omega(-z)+\frac{1}{N^3}\sum_{k,m}\frac{1}{(\lambda_m+\lambda_k)(z-\lambda_k)}\left(\frac{1}{z-\lambda_k}-\frac{1}{\lambda_m+\lambda_k}\right)+ \\
&\quad +\frac{1}{N^3}\sum_{m,k,l}\frac{1}{(\lambda_m+\lambda_k)(\lambda_l+\lambda_k)(z+\lambda_k)}
\end{aligned}
\tag{3.39}
$$

and

$$
\frac{1}{N}\sum_k\frac{1}{\lambda_k^2\,(z-\lambda_k)}=\frac{\omega(z)-\omega(0)}{z^2}+\frac{1}{z}\frac{1}{N}\sum_k\frac{1}{\lambda_k^2}\,. \tag{3.40}
$$

Equations (3.39) and (3.40) follow from the partial fraction decompositions:

$$
\begin{aligned}
\frac{1}{(z-a)(z-b)(z-c)} &= \frac{1}{(b-a)(c-a)(z-a)}+\frac{1}{(c-b)(a-b)(z-b)}+\frac{1}{(a-c)(b-c)(z-c)} \\
\frac{1}{(z-a)^2(z-b)} &= \frac{1}{(a-b)^2(z-b)}-\frac{1}{(a-b)^2(z-a)}+\frac{1}{(a-b)(z-a)^2}\,.
\end{aligned}
\tag{3.41}
$$

When substituting (3.38), (3.39), (3.40) into (3.30) some of the terms cancel and we are left with

$$
\begin{aligned}
&\Bigg\langle \omega(z)^2\omega(-z)+\frac{1}{N^2}\sum_{k,m}\frac{U'(\lambda_k)}{(\lambda_m+\lambda_k)(z-\lambda_k)}+ \\
&+\frac{1}{N^3}\sum_{m,k,l}\frac{1}{(\lambda_l+\lambda_k)(\lambda_m+\lambda_k)}\left(\frac{1}{z-\lambda_k}-\frac{1}{-z-\lambda_k}\right)- \\
&-\frac{1}{N^2}\left(\frac{\omega(z)-\omega(0)}{4\,z^2}+\frac{1}{4z}\frac{1}{N}\sum_k\lambda_k^{-2}\right)\Bigg\rangle=0\,.
\end{aligned}
\tag{3.42}
$$

As in the quadratic case we can add to this the same equation with z replaced by $-z$. This removes the triple sum and gives

$$
\begin{aligned}
&\Bigg\langle \omega(z)^2\omega(-z)+\omega(z)\omega(-z)^2+\frac{1}{N^2}\sum_{k,m}\frac{U'(\lambda_k)}{\lambda_m+\lambda_k}\left(\frac{1}{z-\lambda_k}-\frac{1}{z+\lambda_k}\right) \\
&-\frac{1}{N^2}\frac{\omega(z)+\omega(-z)-2\omega(0)}{4\,z^2}\Bigg\rangle=0\,.
\end{aligned}
\tag{3.43}
$$

To proceed we introduce the quantity

$$
g(z)=\frac{1}{N^2}\sum_{k,l}\frac{U'(z)-U'(\lambda_k)}{(\lambda_l+\lambda_k)(z-\lambda_k)}\,. \tag{3.44}
$$

As with $f(z)$ in (3.36), for later use we note that $g(z)$ can decomposed into a polynomial part $\bar{g}(z)$ and a pole $1/z$ as

$$
g(z)=\bar{g}(z)-\gamma\cdot\frac{t_{-1}+s/(2N)}{z} \tag{3.45}
$$

where

$$\bar{g}(z)=\frac{1}{N^2}\sum_{k,l}\frac{W'(z)-W'(\lambda_k)}{(\lambda_l+\lambda_k)(z-\lambda_k)}\quad\text{and}\quad\gamma=\frac{1}{N^2}\sum_{k,l}\frac{1}{\lambda_k(\lambda_k+\lambda_l)}=\frac{\omega(0)^2}{2}\,. \tag{3.46}$$

As before, since $W'(z)$ has degree d the large z behavior of the polynomial $\bar{g}(z)$ is (const)$\cdot z^{d-1}$, so that $\bar{g}(z)$ has degree $d{-}1$. The expression $\gamma=\omega(0)^2/2$ has been obtained by evaluating the identity (3.34) at $z{=}0$.

Using (3.32), one finds the following relation for the expectation value of $g(z)$:

$$\langle g(z)\rangle=\left\langle -\frac{1}{N^2}\sum_{k,l}\frac{U'(\lambda_k)}{(\lambda_l+\lambda_k)(z-\lambda_k)}+U'(z)\Big(\omega(z)^2+f(z)-U'(z)\omega(z)\Big)\right\rangle. \tag{3.47}$$

Substituting (3.47) into (3.43) we obtain the *cubic loop equation*:

$$\begin{aligned}&\Big\langle\, \omega(z)^2\omega(-z)-g(z)+U'(z)\big(\omega(z)^2+f(z)-U'(z)\omega(z)\big)+(z\leftrightarrow -z)\,\Big\rangle-\\&-\frac{1}{N^2}\,\frac{\langle\omega(z){+}\omega(-z){-}2\omega(0)\rangle}{4\,z^2}=0\,.\end{aligned} \tag{3.48}$$

3.3.3 Loop equations in terms of contour integrals

The two loop equations (3.33) and (3.48) can be presented in a more compact form when using contour integrals. This form can be obtained as follows. Let γ be a contour that encircles all the eigenvalues but not the point z and not the poles of $\omega(-z)$. One can verify the two identities

$$\begin{aligned}\oint_\gamma\frac{dx}{2\pi i}\frac{2xU'(x)}{z^2{-}x^2}\langle\omega(x)\rangle &= \frac{1}{N}\sum_k\left\langle\frac{2\lambda_kU'(\lambda_k)}{z^2{-}\lambda_k^2}\right\rangle,\\ \oint_\gamma\frac{dx}{2\pi i}\frac{2xU'(x)}{z^2{-}x^2}\langle\omega(x)\omega(-x)\rangle &= \frac{1}{N^2}\sum_{k,l}\left\langle\frac{2\lambda_kU'(\lambda_k)}{z^2{-}\lambda_k^2}\,\frac{-1}{\lambda_k{+}\lambda_l}\right\rangle\end{aligned} \tag{3.49}$$

and insert them into the equations (3.33) and (3.43). This results in the following constraints, which are equivalent with the loop equations (3.33) and (3.48):

$$\begin{aligned}\langle\omega(z)^2+\omega(z)\omega(-z)+\omega(-z)^2\rangle &= \oint_\gamma\frac{dx}{2\pi i}\,\frac{2xU'(x)}{z^2{-}x^2}\langle\omega(x)\rangle,\\ \langle\omega(z)^2\omega(-z)+\omega(z)\omega(-z)^2\rangle-&\\ -\frac{1}{N^2}\frac{\langle\omega(z){+}\omega(-z){-}2\omega(0)\rangle}{4\,z^2} &= \oint_\gamma\frac{dx}{2\pi i}\,\frac{2xU'(x)}{z^2{-}x^2}\langle\omega(x)\omega(-x)\rangle.\end{aligned}$$

3.4 The large-N Riemann surface

In this section we shall take the large-N limit of the loop equations for the orientifold with symmetric and antisymmetric matter, thus obtaining two polynomial constraints on the planar limit ω_0 of the averaged resolvent,[8] which is the leading term in the large-N

[8]These constraints also follow from the loop equations of the A_2-quiver, which are given in appendix A.2.1.

expansion:

$$\langle \omega(z) \rangle = \omega_0(z) + \frac{1}{N}\omega_1 + O\left(\frac{1}{N^2}\right). \tag{3.50}$$

These large N constraints on ω_0 define a Riemann surface, similar to what happens in the much better studied case of one-matrix models. The main difference is that the Riemann surface governing our models is not hyperelliptic. Rather, it will turn out to be a triple cover of the z-plane, one of whose branches gives $\omega_0(z)$.

3.4.1 The loop equations at large-N

The large-N limit of the loop equations (3.33) and (3.48) gives:

$$\begin{aligned} \omega_0(z)^2+\omega_0(z)\omega_0(-z)+\omega_0(-z)^2-V'(z)\omega_0(z)-V'(-z)\omega_0(-z)+f(z)+f(-z) &= 0 \\ \omega_0(z)^2\omega_0(-z) - g(z) + V'(z)\big(\omega_0(z)^2 + f(z) - V'(z)\omega_0(z)\big) + (z \leftrightarrow -z) &= 0\,. \end{aligned} \tag{3.51}$$

It is convenient to introduce the shifted variable $u(z)$ through the relations:

$$\omega_0(z) = u(z) - t(z)\,, \qquad \omega_0(-z) = u(-z) - t(-z)\,, \tag{3.52}$$

where

$$t(z) = \frac{-2V'(z) + V'(-z)}{3}\,, \qquad t(-z) = \frac{-2V'(-z) + V'(z)}{3}\,. \tag{3.53}$$

Under this translation, equations (3.51) become:

$$\begin{aligned} u(z)^2 + u(z)u(-z) + u(-z)^2 &= p(z) \\ u(z)^2u(-z) + u(z)u(-z)^2 &= -q(z)\,, \end{aligned} \tag{3.54}$$

where:

$$\begin{aligned} p(z) &= t(z)^2 + t(z)t(-z) + t(-z)^2 - f(z) - f(-z) \\ q(z) &= -t(z)t(-z)\,[t(z) + t(-z)] + t(z)f(-z) + t(-z)f(z) - g(z) - g(-z)\,. \end{aligned} \tag{3.55}$$

Equations (3.54) are the Viete relations for the roots $u_0(z) := u(z), u_1(z) := u(-z)$ and $u_2(z) := -u_0(z) - u_1(z)$ of the cubic:

$$\prod_{i=0}^{2}(u(z) - u_i(z)) \;=\; u^3 - p(z)u - q(z) = 0\,. \tag{3.56}$$

Therefore, the quantities $u_0(z), u_1(z)$ and $u_2(z)$ are the three branches of the affine curve (3.56), when the latter is viewed as a triple cover of the complex z-plane.

Let us index the branches such that:

$$\omega_0(z) = u_0(z) - t(z) \quad \text{and} \quad \omega_0(-z) = u_1(z) - t(-z)\,. \tag{3.57}$$

It is clear from (3.55) that the functions $p(z)$ and $q(z)$ are even. Hence the curve (3.56) admits the automorphism:

$$\nu(z, u) = (-z, u)\,, \tag{3.58}$$

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which permutes the sheets u_0 and u_1, while stabilizing the third sheet. Using (3.55), our curve can be written:

$$\big(u-t(z)\big)\big(u-t(-z)\big)\big(u+t(z)+t(-z)\big)+[f(z)+f(-z)]\,u+ \\ +[2g_{ev}(z)-t(z)f(-z)-t(-z)f(z)]=0\,, \tag{3.59}$$

where:

$$g_{ev}(z):=\frac{1}{2}\,[g(z)+g(-z)]=\frac{1}{2}\,[\bar g(z)+\bar g(-z)] \tag{3.60}$$

is the even part of g. For $f=g\equiv 0$, the defining equation reduces to:

$$(u-t_0(z))(u-t_1(z))(u-t_2(z))=0\,, \tag{3.61}$$

where $t_0(z):=t(z), t_1(z):=t(-z), t_2(z)=-t_0(z)-t_1(z)$. Correspondingly, the branches are given by $u_j=t_j$ in this limit. The general curve (3.59) is a deformation of (3.61), parameterized by the coefficients of f and g_{ev}.

The number of independent coefficients in f and g_{ev} is constrained by the matrix model. Recall from the discussion of equations (3.37) and (3.46) that the polynomials $\bar f(z)$ and $\bar g(z)$ have degree $d{-}1$ if the polynomial part $W(x)$ of the potential has degree $d{+}1$. Thus $\bar f(z)$ depends on d complex parameters while the even polynomial g_{ev} depends on $\delta{+}1$ parameters, where

$$\delta=\left[\frac{d-1}{2}\right]. \tag{3.62}$$

In the large-N limit the function $f(z)$ in (3.35) has the form $f(z)=\bar f(z)-\phi t_{-1}/z$, so it depends on the coefficients of $\bar f(z)$ as well as on ϕ, if the logarithmic term is present in the potential. Altogether, we have:

$$\#(\text{coefficients in } f,\, g_{ev}\,)=\begin{cases} d+\delta+1\,; & t_{-1}=0 \quad a) \\ d+\delta+2\,; & t_{-1}\neq 0 \quad b)\end{cases} \tag{3.63}$$

3.5 The Riemann surface in the absence of a logarithmic deformation

3.5.1 General description and parameter count

In the absence of a logarithmic deformation ($t_{-1}=0$), the large N algebraic curve is given by (3.56) with $V(z)=W(z)$. We shall start by counting its parameters and periods. For this, we must describe the branching structure of this curve and of its deformations. For a generic deformation (f,g), we shall find $d+\delta+1$ independent periods, in agreement with the parameter count for g_{ev} and f performed in the previous subsection.

Let us start with the classical curve (3.61) and analyze its branching. Writing $W'(z)=\sum_{m=0}^{d}t_m z^m$, we have:

$$W'_{odd}(z):=\frac{1}{2}[W'(z)-W'(-z)]=\sum_{k=0}^{\delta}t_{2k+1}z^{2k+1}=zv\left(z^2\right), \tag{3.64}$$

where we defined $v(x):=\sum_{k=0}^{\delta}t_{2k+1}x^k$. Consider the factorizations:

$$W'(z)=t_d\left(z-z_1^+\right)\cdots(z-z_d^+)\rightarrow W'(-z)=t_d(-1)^d(z-z_1^-)\cdots(z-z_d^-) \tag{3.65}$$

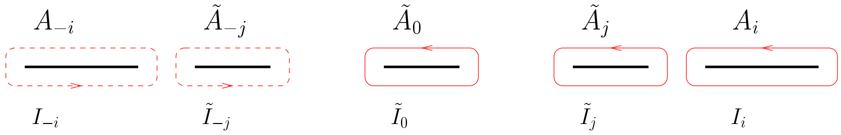

Figure 1: The cycles A_i and $\tilde{A}_j$. Solid lines lie on the sheet $u_0(z)$, while dashed lines lie on $u_1(z)$. Under the automorphism ν the cycles transform as $\nu(A_i) = A_{-i}$, $\nu(\tilde{A}_j) = \tilde{A}_{-j}$ and $\nu(\tilde{A}_0) = -\tilde{A}_0$.

with $z_i^- := -z_i^+$ and

$$v(x) = t_{2\delta+1}(x - x_1) \cdots (x - x_\delta)\,. \tag{3.66}$$

The second relation implies the factorization:

$$W'_{odd}(z) = t_{2\delta+1} z \left(z^2 - x_1\right) \cdots \left(z^2 - x_\delta\right) = t_{2\delta+1} z \left(z - \tilde{z}_1^+\right)\left(z - \tilde{z}_1^-\right) \cdots \left(z - \tilde{z}_\delta^+\right)\left(z - \tilde{z}_\delta^-\right), \tag{3.67}$$

where $\tilde{z}_j^\pm = \pm\sqrt{x_j}$.

Then the curve (3.61) has ordinary double points at those values of z where $t_i(z) = t_j(z)$ for some $0 \le i < j \le 2$. Let us assume that all roots $z_i^+, z_i^-, \tilde{z}_j^+$ and $\tilde{z}_j^-$ are mutually distinct and nonzero (this is the generic case). Then:

$$\begin{aligned} t_0(z) &= t_2(z) \Leftrightarrow W'(z) = 0 \Leftrightarrow z \in \left\{z_1^+ \dots z_d^+\right\} \\ t_1(z) &= t_2(z) \Leftrightarrow W'(-z) = 0 \Leftrightarrow z \in \left\{z_1^- \dots z_d^-\right\} \\ t_0(z) &= t_1(z) \Leftrightarrow W'_{odd}(z) = 0 \Leftrightarrow z \in \left\{\tilde{z}_1^- \dots \tilde{z}_\delta^-, 0, \tilde{z}_1^+ \dots \tilde{z}_\delta^+\right\}. \end{aligned} \tag{3.68}$$

When turning on generic deformations f, g to reach the curve (3.56), all of these double points will split into cuts:

$$\begin{aligned} z_i^+ &\to [a_i, b_i] : = I_i\,, \quad z_i^- \to [-b_i, -a_i] := I_{-i}\,, \quad i = 1, \dots, d \\ \tilde{z}_j^+ &\to [\tilde{a}_j, \tilde{b}_j] : = \tilde{I}_j\,, \quad \tilde{z}_j^- \to [-\tilde{b}_j, -\tilde{a}_j] := \tilde{I}_{-j}\,, \quad j = 1, \dots, \delta \\ \tilde{z}_0 &\to [\tilde{a}_0, \tilde{b}_0] : = \tilde{I}_0\,, \quad \text{with} \quad \tilde{a}_0 = -\tilde{b}_0\,. \end{aligned} \tag{3.69}$$

This splitting is symmetric since the allowed deformations must preserve the symmetry of our curve. We obtain $2d$ cuts of type I_i and $2\delta + 1$ cuts of type $\tilde{I}_j$.

The deformed curve (3.56) has a multiple point at $z = \infty$. It follows that an appropriate deformation of its normalization will have two supplementary cuts. Applying the Hurwitz formula for this normalized and deformed curve, we obtain its genus g:

$$g = \frac{1}{2}(4\delta + 4d + 6) - 3 + 1 = 2(d + \delta) + 1\,. \tag{3.70}$$

This is one unit greater than the arithmetic genus of the curve (3.56).

Choose cycles A_i, $\tilde{A}_j$ around the cuts I_i, $\tilde{I}_j$ as indicated in figure 1. Due to the $\mathbb{Z}_2$ invariant perturbation of the cuts and the induced action $\nu^*(\lambda) = -\lambda$ of (3.58) on the meromorphic differential

$$\lambda = \frac{1}{2\pi i} u\, dz\,, \tag{3.71}$$

the periods integrals of this 'regularized' curve along $A_{-i}, \tilde{A}_{-j}$ are minus the ones along $A_i, \tilde{A}_j$ for $i = 1 \dots d$ and $j = 1 \dots \delta$, while $\int_{\tilde{A}_0} \lambda$ is invariant. Therefore, we can choose $\int_{A_i} \lambda$ and $\int_{\tilde{A}_j} \lambda$ (with $i = 1 \dots d$ and $j = 0 \dots \delta$) to be independent periods. Hence their number agrees with the parameter count for g_{ev} and f. In (3.69) we assumed a generic situation, which can be justified from the double cover. In that case, the existence of $3d$ independent 3-cycles in the Calabi-Yau geometry has been established [36]. By the classical results of Tian and Todorov [44, 45], this is equivalent to $3d$ unobstructed complex structure parameters, which descend to the Riemann surface. The argument above can be viewed as establishing consistency of the $\mathbb{Z}_2$ projection on the parameters and periods of the Riemann surface.

3.5.2 Physical meaning of the cuts

We next discuss the interpretation of the cuts in terms of the planar eigenvalue distribution of the matrix model. Remember that $\omega_0(z) = \int d\lambda \frac{\rho(\lambda)}{z-\lambda}$. From (3.57) we see that the cuts of $\omega_0(z)$ coincide with those of $u_0(z)$. The fact that u_0 has branch cuts along I_i and $\tilde{I}_j$ (with $i = 1 \dots d$ and $j = -\delta \dots \delta$) requires that $\rho(\lambda)$ be non-vanishing along *each* of these cuts. This implies that $\omega_0(-z)$ will have cuts on the reflected loci I_{-i} and $\tilde{I}_{-j}$. To find the matrix model meaning of these cuts, consider the analytic function:

$$\begin{aligned}\kappa(z) &:= u_0(z) - u_2(z) = 2u_0(z) + u_0(-z) = 2\omega_0(z) + \omega_0(-z) - W'(z) \\ &= \int d\lambda \rho(\lambda) \left[\frac{2}{z-\lambda} - \frac{1}{z+\lambda}\right] - W'(z)\,,\end{aligned} \tag{3.72}$$

where we used relations (3.52) and (3.53), which imply:

$$2t(z) + t(-z) = -W'(z)\,. \tag{3.73}$$

Similarly, we consider:

$$\begin{aligned}\tau(z) &:= u_0(z) - u_1(z) = u_0(z) - u_0(-z) = \omega_0(z) - \omega_0(-z) - [W'(z) - W'(-z)] \\ &= \int d\lambda \rho(\lambda) \left[\frac{1}{z-\lambda} + \frac{1}{z+\lambda}\right] - [W'(z) - W'(-z)]\,,\end{aligned} \tag{3.74}$$

where again we used (3.52) and (3.53).

The definition of the cuts I_i and $\tilde{I}_j$ implies (see figure 2):

$$\begin{aligned}u_0(\lambda \pm i0) &= u_2(\lambda \mp i0) \quad \text{for} \quad \lambda \in I_i \quad \text{with} \quad i = 1 \dots d \\ u_1(\lambda \pm i0) &= u_2(\lambda \mp i0) \quad \text{for} \quad \lambda \in I_{-i} \quad \text{with} \quad i = 1 \dots d \\ u_0(\lambda \pm i0) &= u_1(\lambda \mp i0) \quad \text{for} \quad \lambda \in \tilde{I}_j \quad \text{with} \quad j = -\delta \dots \delta\,.\end{aligned} \tag{3.75}$$

Therefore, we have:

$$\begin{aligned}\mathcal{P}(\kappa(\lambda)) &= 0 \quad \text{for} \quad \lambda \in I_i \quad \text{with} \quad i = 1 \dots d \\ \mathcal{P}(\kappa(-\lambda)) &= 0 \quad \text{for} \quad \lambda \in I_{-i} \quad \text{with} \quad i = 1 \dots d \\ \mathcal{P}(\tau(\lambda)) &= 0 \quad \text{for} \quad \lambda \in \tilde{I}_j \quad \text{with} \quad j = -\delta \dots \delta\,.\end{aligned} \tag{3.76}$$

Writing these principal values using (3.72) and (3.74), we find:

$$\int d\lambda'\rho(\lambda')\left[\frac{2}{\lambda-\lambda'}-\frac{1}{\lambda+\lambda'}\right] = W'(\lambda)\,, \quad \lambda\in I_i\,, \tag{3.77}$$

$$\int d\lambda'\rho(\lambda')\left[\frac{2}{\lambda+\lambda'}-\frac{1}{\lambda-\lambda'}\right] = -W'(-\lambda)\,, \quad \lambda\in I_{-i}\,, \tag{3.78}$$

$$\int d\lambda'\rho(\lambda')\left[\frac{1}{\lambda-\lambda'}+\frac{1}{\lambda+\lambda'}\right] = W'(\lambda)-W'(-\lambda)\,, \quad \lambda\in\tilde{I}_j\,. \tag{3.79}$$

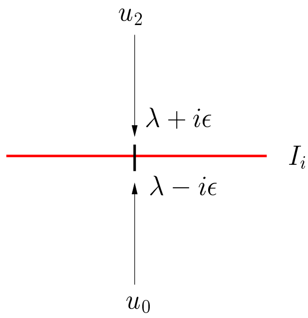

Figure 2: The values of $u_0(\lambda - i0)$ and $u_2(\lambda + i0)$ agree along a cut I_i, since the latter connects the branches u_0 and u_2. A similar argument holds for the other types of cuts.

Notice that the second equation is equivalent with the first under the substitution $\lambda \to -\lambda$ and therefore contains no new information.

Assuming that all cuts are disjoint from each other, the analytic function u_2 must be continuous across $\tilde{I}_j$ (since the cuts of u_2 are I_j and I_{-j}). Therefore, we have the following relations for $\lambda\in\tilde{I}_j$:

$$\begin{aligned} u_2(\lambda+i0) = u_2(\lambda-i0) &\Longleftrightarrow u_0(\lambda+i0)+u_1(\lambda+i0) = u_0(\lambda-i0)+u_1(\lambda-i0) \\ &\Longleftrightarrow u_0(\lambda+i0)-u_0(\lambda-i0) = u_1(\lambda-i0)-u_1(\lambda+i0)\,. \end{aligned} \tag{3.80}$$

Using $u_0(z) = u_1(-z)$ in the right hand side gives:

$$\rho(\lambda) = \rho(-\lambda)\,, \qquad \lambda\in\tilde{I}_j\,, \tag{3.81}$$

where we also used the identity:

$$u_0(\lambda-i0)-u_0(\lambda+i0) = 2\pi i\rho(\lambda)\,, \qquad \lambda\in\mathbb{R}\,. \tag{3.82}$$

Thus ρ takes symmetric values along the cuts $\tilde{I}_j$ and $\tilde{I}_{-j}$ (in particular, $\rho(\lambda)$ is symmetric along $\tilde{I}_0$).

Finally, we can identify the filling fractions of the matrix model:

$$\begin{aligned} S_i &= \int_{A_i}\lambda\,, \quad \text{for} \quad i=1\dots d\,, \\ \tilde{S}_j &= 2\int_{\tilde{A}_j}\lambda\,, \quad \text{for} \quad j=1\dots\delta \\ \tilde{S}_0 &= \int_{\tilde{A}_0}\lambda\,. \end{aligned} \tag{3.83}$$

The factors of two in the second equation follows from the fact that $\tilde{I}_j$ and $\tilde{I}_{-j}$ support symmetric distributions of eigenvalues for $j\neq 0$.

From the gauge theory point of view, we can interpret (3.77) as the quantum correction to the classical vacuum configuration (2.8) and (3.79) as the correction to (2.7). The different filling fractions correspond to the different types of vacua we found in section 2.1. In particular, a fraction $\tilde{S}_j$ with $j\neq 0$ corresponds to a vacuum in which the final $\mathrm{U}(N_j)$ gauge group is embedded diagonally, while $\tilde{S}_0$ corresponds to the vacuum with an orthogonal or symplectic unbroken gauge group. Of course, up to now we imposed the strict large-N limit and therefore the curve can not distinguish between the two types of orientifolds. In section 5 we will argue that this difference can be accounted for by including a logarithmic term in the potential V of the model. Let us therefore study this situation next.

3.6 Riemann surface in the presence of a logarithmic interaction

In this subsection, we analyze the Riemann surface (3.56) in the presence of a logarithmic term $t_{-1} \ln z$ in the potential V. We shall show that turning on such interactions produces supplementary cuts, which correspond to eigenvalues accumulating along new loci. The purpose of the present section is to identify these novel cuts for small values of t_{-1} — this will important in our discussion of the first order loop equations in section 5.

To understand the generic situation, first note that $p(z)$ and $q(z)$ acquire poles at the origin for $t_{-1}\neq 0$. To display this, we decompose $t(z) = \bar{t}(z) - \frac{t_{-1}}{z}$, with $\bar{t}(z) := \frac{-2W'(z)+W'(-z)}{3}$ a polynomial. Further, we define polynomials $\bar{p}(z)$ and $\bar{q}(z)$ by substituting $\bar{f}$, $\bar{g}$, $\bar{t}$ in (3.55). This results in:

$$\begin{aligned} p(z) &= \bar{p}(z) - \frac{t_{-1}}{z}\big(\bar{t}(z)-\bar{t}(-z)\big) + \frac{t_{-1}^2}{z^2} \\ q(z) &= \bar{q}(z) + \frac{t_{-1}}{z}\big(\bar{f}(z)-\bar{f}(-z) + [\bar{t}(z)-\bar{t}(-z)][\phi-\bar{t}(z)-\bar{t}(-z)]\big) - \\ &\quad - \frac{t_{-1}^2}{z^2}\big(2\phi - \bar{t}(z) - \bar{t}(-z)\big)\,, \end{aligned} \tag{3.84}$$

where we used (3.36) and (3.46) and absorbed the s dependence in t_{-1}. To write (3.56) in polynomial form, we multiply the equation by z^3 and define a new variable $y := zu$:

$$y^3 - P(z)y - zQ(z) = (y - y_0(z))(y - y_1(z))(y - y_2(z)) = 0\,, \tag{3.85}$$

where $y_1(z) = -y_0(-z)$, $y_2(z) = -y_0(z) - y_1(z)$ and $P(z) = z^2 p(z)$ and $Q(z) := z^2 q(z)$ are even polynomials. This curve admits the symmetry:

$$(z, y) \to (-z, -y)\,. \tag{3.86}$$

We shall view (3.85) as a deformation of the curve:

$$\bar{y}^3 - z^2\bar{p}(z)\bar{y} - z^3\bar{q}(z) = (\bar{y} - \bar{y}_0(z))(\bar{y} - \bar{y}_1(z))(\bar{y} - \bar{y}_2(z)) = 0\,, \tag{3.87}$$

which is obtained from:

$$u^3 - \bar{p}(z)u - \bar{q}(z) = (u - \bar{u}_0(z))(u - \bar{u}_1(z))(u - \bar{u}_2(z)) = 0\,, \tag{3.88}$$

where $\bar{u}_2(z) = -\bar{u}_0(z) - \bar{u}_1(z)$ and $\bar{u}_1(z) = \bar{u}_0(-z)$, by performing the birational transformation $\bar{y} = zu$.

We are interested in the generic behavior of (3.85) as t_{-1} approaches zero. In this limit, the curve degenerates to (3.87), whose first two branches $\bar{y}_0$ and $\bar{y}_1$ are tangent in a cusp at $\bar{y} = z = 0$, where they also meet the third branch $\bar{y}_2$ (figure 3).

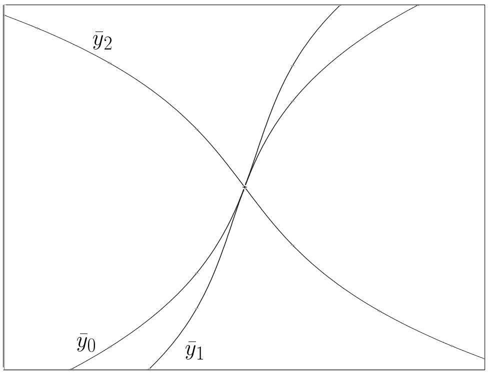

Figure 3: Branching of the curve (3.85) at the origin for $t_{-1} = 0$.

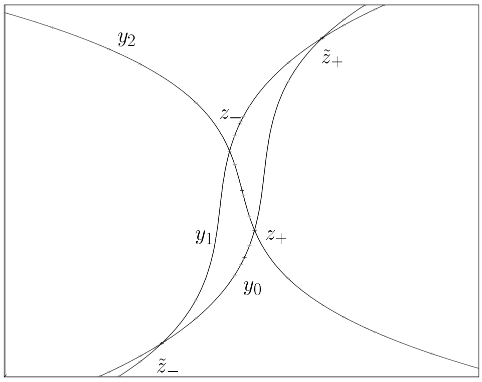

Figure 4: Deformed curve after addition of a logarithmic term in the matrix model potential, in the birationally equivalent coordinates (y, z).

The geometry of (3.85) for a small value of t_{-1} is described in figure 4. The logarithmic deformation of the model generates four new branch points of the birationally transformed curve, which are indexed by $z_\pm$ and $\tilde{z}_\pm$. These are symmetric with respect to the origin, namely $z_- = -z_+$ and $\tilde{z}_- = -\tilde{z}_+$.

From this geometry, one finds that the cut $\tilde{I}_0$ splits into two new cuts $\tilde{I}_0^+ = [\tilde{z}_+, \tilde{b}_0]$, and $\tilde{I}_0^- = [\tilde{a}_0, \tilde{z}_-]$, which are distributed symmetrically with respect to the origin. Both of these cuts connect the branches 0 and 1. One also finds two new cuts $I_0 = (-\infty, z_+)$ and $-I_0 = (z_-, +\infty)$, which connect the pairs of branches $(0, 2)$ and $(1, 2)$ respectively. As in [5], we shall 'regularize' our curve by choosing a large $\Lambda > 0$ and replacing these cuts with $I_0 = (-\Lambda, z_+)$ and $-I_0 = (z_-, +\Lambda)$.

These cuts produce new A-type cycles A_0, $\tilde{A}_0^\pm$ (and associated B-cycles), which project onto the z-plane to the curves $\gamma_0, \tilde{\gamma}_0^\pm$, see figure 5. The homology classes $\tilde{A}_0^\pm$ are interchanged by the symmetry $z \to -z$, which also interchanges the homology class of A_0 with that of $-A_0$.

The cuts I_0 and $\tilde{I}_0^\pm$ give the A-type periods:

$$\begin{aligned} \oint_{\gamma_0} \frac{dz}{2\pi i}\omega_0(z) &= \oint_{A_0} \frac{dz}{2\pi i}\omega(z)\,, \\ \oint_{\tilde{\gamma}_0^\pm} \frac{dz}{2\pi i}\omega_0(z) &= \oint_{\tilde{A}_0^\pm} \frac{dz}{2\pi i}\omega(z)\,. \end{aligned} \tag{3.89}$$

The cut $-I_0$ gives the opposite of the A_0-period. The $\mathbb{Z}_2$ symmetry also implies:

$$\oint_{\tilde{\gamma}_0^-} \frac{dz}{2\pi i}\omega_0(z) = \oint_{\tilde{\gamma}_0^+} \frac{dz}{2\pi i}\omega_0(z)\,. \tag{3.90}$$

The presence of new cuts for the log-deformed model shows that eigenvalues can accumulate along I_0 and $\tilde{I}_0^\pm$ in the large-N limit (no eigenvalues accumulate on $-I_0$ since this is not a cut of ω_0). Accordingly, the microcanonical ensemble of the log-deformed model

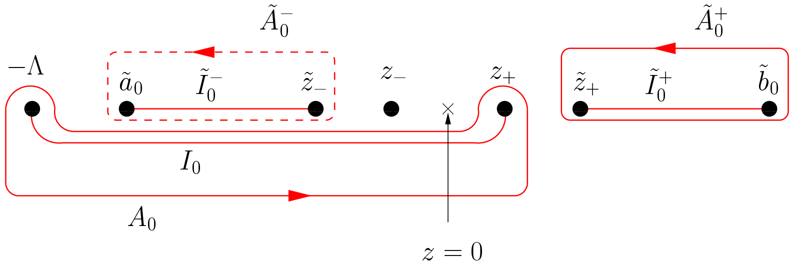

Figure 5: The new cuts of u_0 generated by a logarithmic term in the potential (viewed as cuts of the birationally transformed curve (3.85)). The cut $-I_0$ corresponds to branching of u_1 with u_2, and consequently it is not shown. The figure also indicates the new cycles A_0, $\tilde{A}_0^{\pm}$ on the Riemann surface. Solid cycles lie on the sheet $u_0(z)$, while the dashed cycle lies on $u_1(z)$. The curves γ_0,$\tilde{\gamma}_0^{\pm}$ mentioned in the text are the projections of A_0 and $\tilde{A}_0^{\pm}$ onto the z-plane.

involves the supplementary constraint:

$$\oint_{\gamma_0} \frac{dz}{2\pi i}\langle\omega(z)\rangle = S_0\,, \tag{3.91}$$

while the constraint $\oint_{\tilde{\gamma}_0} \frac{dz}{2\pi i}\langle\omega(z)\rangle = \tilde{S}_0$ of the undeformed model is replaced by:

$$\oint_{\tilde{\gamma}_0^+} \frac{dz}{2\pi i}\langle\omega(z)\rangle = \frac{1}{2}\tilde{S}_0\,. \tag{3.92}$$

This becomes part of conditions (3.27) for the case $t_{-1} \neq 0$. For a consistent limit as $t_{-1} \to 0$, one takes S_0 to be a function of t_{-1} which tends to zero in that limit. Note also that with the additional constraint (3.91) we have to fix $d + \delta + 2$ periods in the log-deformed model, in agreement with the parameter count in (3.63).

4. Chemical potentials at large-N

In this section, we study the planar limit of our models for the case $t_{-1} = 0$. Our main purpose is to give a *proof* [9] of an appropriate set of special geometry relations. The path we shall follow is based on our construction of the microcanonical ensemble, combined with a modification of an argument due to [46]. Namely, we shall show that the chemical potentials μ reduce in the large-N limit to certain B-type periods Π of our Riemann surface. Then special geometry conditions of the type found in [1] follow from the standard equation (3.26) which expresses the chemical potentials as derivatives of the *microcanonical* partition function with respect to the filling fractions. In particular, equation (3.26) is the

[9]The original paper [1] gives a beautiful intuitive argument for the existence of such a relation, without providing a rigorous proof (what is missing is to show that the Riemann surface B-periods indeed equal appropriately defined chemical potentials). A derivation of this relation was later given in [46, 47, 48] (using older results of [49]), though a clear construction of the microcanonical ensemble (which is implicit in that argument) was not given there. In the present paper, we are dealing with the more complicated case of a non-hyperelliptic Riemann surface, which underscores the usefulness of having a clear proof of such relations.

appropriate finite N generalization of the special geometry constraints. We also show that Whitham-type relations of the type found in [46] arise as the planar limit of certain finite N equations which follow naturally in the microcanonical ensemble. For simplicity, we assume $t_{-1} = 0$ for most of the present section. The generalization to the case $t_{-1} \neq 0$ is entirely obvious, but notationally tedious.

Let us hence assume $t_{-1} = 0$. Since $\rho(\lambda)$ in our models develops a nonzero value along the cuts I_i and $\tilde{I}_j$ with $i = 1 \dots d$ and $j = -\delta \dots \delta$, we have $d + 2\delta + 1$ filling fractions. To construct the microcanonical ensemble, we pick intervals Δ_i ($i = 1 \dots d$) and $\tilde{\Delta}_j$ ($j = -\delta \dots \delta$), such that $\tilde{\Delta}_{-j} = -\tilde{\Delta}_j$ for all j. In the large-N limit, we shall assume that the intervals Δ_i and $\tilde{\Delta}_j$ contain the cuts I_i and $\tilde{I}_j$. The symmetry property (3.81) shows that the filling fractions of $\tilde{I}_j$ and $\tilde{I}_{-j}$ are equal, and we shall take $\tilde{S}_0 \dots \tilde{S}_\delta$ to be the independent quantities:

$$\begin{aligned} \int_{I_i} d\lambda \rho_0(\lambda) &= S_i \quad \text{for} \quad i = 1 \dots d \\ \int_{\tilde{I}_j} d\lambda \rho_0(\lambda) &= \frac{1}{2}\tilde{S}_j \quad \text{for} \quad j = 1 \dots \delta \\ \int_{\tilde{I}_0} d\lambda \rho_0(\lambda) &= \tilde{S}_0 \,. \end{aligned} \tag{4.1}$$

Then the $d+\delta+1$ constraints (4.1) completely fix the $d+\delta+1$ deformations of (3.56) encoded by the polynomials f and g_{ev}. In the (grand) canonical ensemble, this is implemented by introducing chemical potentials μ_i and $\tilde{\mu}_j$ with $i = 1 \dots d$ and $j = 0 \dots \delta$.

The planar (grand) canonical generating function reads:

$$\begin{aligned} \mathcal{F}_0(t,\mu) = \int d\lambda W(\lambda)\rho_0(\lambda) - \frac{1}{2}\int d\lambda \int d\lambda' \mathcal{K}(\lambda,\lambda')\rho_0(\lambda)\rho_0(\lambda') + \\ + \sum_{i=1}^{d} \mu_i S_i + \sum_{j=0}^{\delta} \tilde{\mu}_j \tilde{S}_j \end{aligned} \tag{4.2}$$

where

$$\mathcal{K}(\lambda,\lambda') = 2\ln|\lambda - \lambda'| - \ln|\lambda + \lambda'| \,. \tag{4.3}$$

This gives the planar limit of the microcanonical generating function:

$$F_0(t,S) = \frac{1}{2}\int d\lambda \int d\lambda' \mathcal{K}(\lambda,\lambda')\rho_0(\lambda)\rho_0(\lambda') - \int d\lambda W(\lambda)\rho_0(\lambda) \,, \tag{4.4}$$

with the constraints (4.1).

Observation. Remembering relation (3.81), one finds the functional derivatives:

$$\begin{aligned} \frac{\delta F_0}{\delta \rho(\lambda)} &= \int d\lambda' \mathcal{K}(\lambda,\lambda')\rho_0(\lambda') - W(\lambda) \,, \quad \lambda \in I := \cup_{i=1\dots d} I_i \\ \frac{\delta F_0}{\delta \rho(\lambda)} &= \int d\lambda' \tilde{\mathcal{K}}(\lambda,\lambda')\rho_0(\lambda') - W(\lambda) - W(-\lambda) \,, \quad \lambda \in \tilde{I} := \cup_{j=-\delta\dots\delta} \tilde{I}_j \,, \end{aligned} \tag{4.5}$$

where:

$$\tilde{\mathcal{K}}(\lambda,\lambda') = \mathcal{K}(\lambda,\lambda') + \mathcal{K}(-\lambda,\lambda') = \ln|\lambda+\lambda'| + \ln|\lambda-\lambda'| \,. \tag{4.6}$$

Note that equating (4.5) to constants and further differentiating with respect to λ gives the planar equations of motion (3.77), (3.78), (3.79). This observation can be viewed as an intuitive justification for the rigorous procedure discussed below.

4.1 The primitive of κ along the real axis

To extract the large-N chemical potentials, we shall be interested in the 'restriction' of the function κ of subsection 3.5.2 along the real axis, which we define by:

$$\kappa_p(\lambda) = \frac{1}{2}\left[\kappa(\lambda+i0) + \kappa(\lambda-i0)\right] \tag{4.7}$$

for any real λ. If λ is a real value lying outside the union of I_i, then $\kappa_p(\lambda)$ equals $\kappa(\lambda)$, the quantity obtained by substituting λ for z in (3.72). Taking the principal value of (3.72) along the real axis gives:

$$\kappa_p(\lambda) := \int d\lambda' \rho_0(\lambda') K(\lambda,\lambda') - W'(\lambda)\,, \tag{4.8}$$

where $K(z,z') := \frac{2}{z-z'} - \frac{1}{z+z'}$. Consider now the function $\phi : \mathbb{R} \to \mathbb{C}$ defined through:

$$\phi(\lambda) := \int d\lambda' \mathcal{K}(\lambda,\lambda') \rho_0(\lambda') - W(\lambda)\,. \tag{4.9}$$

Noticing that $K(\lambda,\lambda') = \frac{\partial}{\partial\lambda}\mathcal{K}(\lambda,\lambda')$ shows that ϕ is a primitive of κ_p:

$$\frac{d\phi}{d\lambda} = \kappa_p\,. \tag{4.10}$$

As discussed in the previous section, the equations of motion (3.28) amount to the requirement that κ_p vanishes along each of the intervals I_i:

$$\kappa_p(\lambda) = 0 \quad \text{for} \quad \lambda \in I = \cup_{i=1}^{d} I_i\,. \tag{4.11}$$

This means that ϕ is constant along each of these intervals:

$$\phi(\lambda) = \xi_i = \text{ constant for } \lambda \in I_i\,. \tag{4.12}$$

Then the jump in the value of κ_p between consecutive cuts can be obtained by integrating (4.10) between their endpoints:

$$\xi_{i+1} - \xi_i = \int_{b_i}^{a_{i+1}} d\lambda \kappa(\lambda)\,. \tag{4.13}$$

4.2 The primitive of τ along the real axis

An entirely similar discussion can be given for the function $\tau(z)$, whose 'restriction' to the real axis is:

$$\tau_p(\lambda) = \frac{1}{2}\left[\tau(\lambda+i0)+\tau(\lambda-i0)\right] = \int d\lambda' \rho_0(\lambda')\tilde{K}(\lambda,\lambda') - \left[W'(\lambda) - W'(-\lambda)\right], \quad (4.14)$$

where $\tilde{K}(z,z') := \frac{1}{z-z'} - \frac{1}{z+z'}$. Its primitive

$$\psi(\lambda) := \int d\lambda' \tilde{\mathcal{K}}(\lambda,\lambda')\rho_0(\lambda') - W(\lambda) - W(-\lambda) \quad (4.15)$$

can be integrated in between the cuts $\tilde{I}_j$ to give:

$$\tilde{\xi}_{j+1} - \tilde{\xi}_j = \int_{\tilde{b}_j}^{\tilde{a}_{j+1}} d\lambda \tau(\lambda)\,, \quad (4.16)$$

where $\tilde{\xi}_j$ is the constant value of ψ along $\tilde{I}_j$ (as required by the second planar equation of motion):

$$\tau_p(\lambda) = 0 \quad \text{for} \quad \lambda \in \tilde{I} = \cup_{j=-\delta\ldots\delta}\tilde{I}_j\,. \quad (4.17)$$

4.3 The large-N chemical potentials

Differentiating (4.4) with respect to S_i for some $i < d$ and using relation (4.9) gives:

$$\mu_i^{(0)} - \mu_d^{(0)} = \frac{\partial}{\partial S_i}F_0(t,S) = \int_{I\cup\tilde{I}} d\lambda \frac{\partial\rho_0(\lambda)}{\partial S_i}\phi(\lambda) = \xi_i - \xi_d\,, \quad (4.18)$$

where we took the dependent filling fraction to be S_d.

To arrive at the last equality, we used equation (4.12) and the constraints (4.1) and (3.25). Relation (4.18) shows that the chemical potentials associated with I_i coincide with the quantities ξ_i in the planar limit, up to a common additive constant. Using relation (4.13), we obtain:

$$\mu_{i+1}^{(0)} - \mu_i^{(0)} = \int_{b_i}^{a_{i+1}} d\lambda \kappa(\lambda)\,. \quad (4.19)$$

A similar argument for the differential with respect to $\tilde{S}_j$ (using relation (4.15) and the last two constraints in (4.1)) gives $\tilde{\mu}_i^{(0)} = \tilde{\xi}_i$ and:

$$\tilde{\mu}_{j+1}^{(0)} - \tilde{\mu}_j^{(0)} = \int_{\tilde{b}_j}^{\tilde{a}_{j+1}} d\lambda \tau(\lambda)\,. \quad (4.20)$$

4.4 Geometric expression for the large-N chemical potentials

Together with (3.72), equation (4.19) gives:

$$\mu_i^{(0)} - \mu_{i+1}^{(0)} = \int_{b_i}^{a_{i+1}} dz[u_2(z) - u_0(z)] = \oint_{\bar{B}_i} dz u(z)\,, \quad (4.21)$$

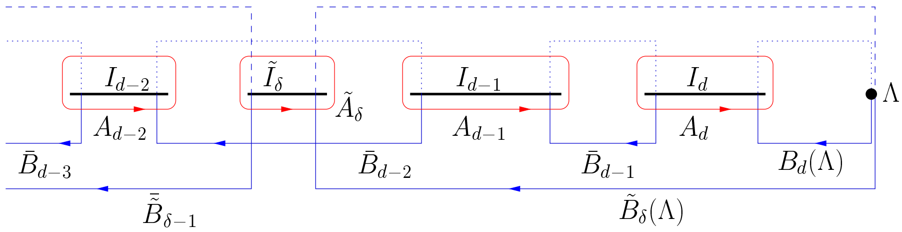

Figure 6: Choice of B-cycles on the large-N Riemann surface. We only indicate a few cuts close to the point Λ. The cycles $\bar{B}_i, \bar{\tilde{B}}_j$ and $B_d(\Lambda), \tilde{B}_\delta(\Lambda)$ are defined such that, when crossing a cut going upwards along these cycles, one moves from the branch u_0 to the branch u_1 (drawn dashed) or u_2 (dotted) respectively. With the orientation of the Riemann surface induced by its complex structure, and with the indexing α of cuts explained the text, this implies the intersections $A_\alpha \cap \bar{B}_\alpha = +1$ for the re-indexed cycles (note that the cycles A_α and $\bar{B}_\alpha$ intersect in a single point, which lies on the branch u_0).

where $\bar{B}_i$ are cycles on the large-N Riemann surface chosen as explained in figure 6. Similarly, we find:

$$\tilde{\mu}_j^{(0)} - \tilde{\mu}_{j+1}^{(0)} = \int_{\tilde{b}_j}^{\tilde{a}_{j+1}} dz[u_1(z) - u_0(z)] = \oint_{\bar{\tilde{B}}_j} dz u(z)\,, \tag{4.22}$$

with $\bar{\tilde{B}}_j$ chosen as in figure 6.

Consider the cycles $\hat{B}_i = \sum_{k=i}^{d-1} \bar{B}_k$ for all $i = 1 \ldots d-1$. Then (4.21) implies:

$$\mu_i^{(0)} = \mu_d^{(0)} + \oint_{\hat{B}_i} dz u(z) \quad \text{for} \quad i = 1 \ldots d-1\,. \tag{4.23}$$

The quantity $\mu_d^{(0)}$ is undetermined and can be fixed arbitrarily. Following [1], we take $\mu_d^{(0)} = \oint_{B_d} dz u(z)$, where Λ is a point close to $+\infty$ and $B_d(\Lambda)$ is a path shown in figure 6. Defining $B_i(\Lambda) = \hat{B}_i + B_d(\Lambda)$ for all $i = 1 \ldots d-1$, equation (4.23) gives:

$$\mu_i^{(0)} = \Pi_i \quad \text{for} \quad i = 1 \ldots d\,, \tag{4.24}$$

with:

$$\Pi_i := \int_{B_i} dz u(z)\,. \tag{4.25}$$

A similar construction for the cuts $\tilde{I}_j$ gives:

$$\tilde{\mu}_j^{(0)} = \tilde{\Pi}_j \quad \text{for} \quad j = 0 \ldots \delta\,, \tag{4.26}$$

where:

$$\tilde{\Pi}_j := \int_{\tilde{B}_j} dz u(z)\,, \tag{4.27}$$

with cycles $\tilde{B}_j$ chosen analogously (figure 6):

$$\tilde{B}_j = \tilde{B}_\delta(\Lambda) + \sum_{k=j}^{\delta-1} \bar{\tilde{B}}_k\,. \tag{4.28}$$

Relations (4.24) and (4.26) show that the chemical potentials μ are the finite N analogues of the periods Π.

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Also remember that the filling fractions can be expressed as periods of udz over the cycles A_α of figure 1:

$$S_\alpha = \oint_{\gamma_\alpha} \frac{dz}{2\pi i}\omega_0(z) = \oint_{\gamma_\alpha} \frac{dz}{2\pi i}u_0(z) = \oint_{A_\alpha} \frac{dz}{2\pi i}u(z)\,. \tag{4.29}$$

In the second equality, we used relation (3.52) and the fact that $t(z)$ is a polynomial.

Defining $q := d + \delta$, let us index the cuts by $\alpha = -q \dots q$ in the order in which the intervals I_i, $\tilde{I}_j$ appear from left to right. We also let $\mathcal{A}_\alpha$ and $\bar{\mathcal{B}}_\alpha$ respectively $\mathcal{B}_\alpha$ be the corresponding cycles $A_i, \tilde{A}_j$ and $\bar{B}_i, \tilde{\bar{B}}_j$ respectively B_i, $\tilde{B}_j$ indexed in this order. It is then clear from figure 6 that $\mathcal{A}_\alpha \cap \bar{\mathcal{B}}_\alpha = -\mathcal{A}_\alpha \cap \bar{\mathcal{B}}_{\alpha-1} = +1$ and $\mathcal{A}_\alpha \cap \bar{\mathcal{B}}_\beta = 0$ if $\beta \neq \alpha, \alpha - 1$. This gives $\mathcal{A}_\alpha \cap \mathcal{B}_\beta = -\mathcal{B}_\beta \cap \mathcal{A}_\alpha = \delta_{\alpha\beta}$. Since we also have $\mathcal{A}_\alpha \cap \mathcal{A}_\beta = \mathcal{B}_\alpha \cap \mathcal{B}_\beta = 0$, it follows that $\mathcal{A}_\alpha, \mathcal{B}_\beta$ have canonical intersection form. Thus we find a canonical system of cycles $\mathcal{A}_\alpha, \mathcal{B}_\alpha$ with $\alpha = -q \dots q$:

$$\mathcal{A}_\alpha \cap \mathcal{B}_\beta = -\mathcal{B}_\alpha \cap \mathcal{A}_\beta = \delta_{\alpha,\beta}\,, \qquad \mathcal{A}_\alpha \cap \mathcal{A}_\beta = \mathcal{B}_\alpha \cap \mathcal{B}_\beta = 0 \quad \text{for all} \quad \alpha = -q \dots q\,. \tag{4.30}$$

4.5 Whitham-type relations

Let us now show how a finite N version of a set of Whitham-type constraints (similar to those found in [46] for the case of one-matrix models) can be extracted from the microcanonical ensemble. For the rest of this subsection, we shall allow for a logarithmic term in the potential— as we shall see in a moment, this has interesting effects on our Whitham-type relations.

Differentiating (3.24) with respect to t_m gives:

$$\frac{\partial}{\partial t_m}F(t,S) = -\frac{\partial}{\partial t_m}\mathcal{F}(t,\mu)|_{\mu_\alpha=\mu_\alpha(t,S)} = l_m\,, \tag{4.31}$$

where we defined the averaged loop operators: $l_m := -\frac{1}{N}\langle tr\partial_{t_m}V(M)\rangle = -\frac{1}{N(m+1)} \times \langle tr(M^{m+1})\rangle$ for $m > 0$ and $l_{-1} := -\frac{1}{N}\langle tr(logM)\rangle$ for $m = -1$. Equations (4.31) supplement relations (3.26). We have:

$$-l_m = \begin{cases} \dfrac{1}{m+1}\displaystyle\int d\lambda\langle\rho(\lambda)\rangle\lambda^{m+1} = \dfrac{1}{m+1}\oint_\gamma \dfrac{dz}{2\pi i}z^{m+1}\langle\omega(z)\rangle \quad \text{if} \quad m > 0 \\ \int d\lambda\langle\rho(\lambda)\rangle \log\lambda = \displaystyle\oint_\gamma \dfrac{dz}{2\pi i}\langle\omega(z)\rangle \ln z \quad \text{if} \quad m = -1\,, \end{cases} \tag{4.32}$$

where γ is a counterclockwise contour that encircles all eigenvalues. Combined with (4.32), equations (4.31) give a finite N version of a set of Whitham-like constraints; these reduce to standard Whitham conditions in the planar limit and for the case $t_{-1} = 0$.

4.5.1 Geometric form of the large-N Whitham-type constraints

For $m > 0$, one can deform the contour γ toward infinity, thus picking up contributions from $z = \infty$. This gives:

$$l_m = \frac{1}{m+1}Res_{\zeta=0}\left[\frac{\langle\omega(1/\zeta)\rangle}{\zeta^{m+3}}\right] \quad \text{for} \quad m > 0\,, \tag{4.33}$$

where $\zeta = 1/z$.

In the large-N limit, one has $\langle\omega(z)\rangle = \omega_0(z)$, with $\omega_0(z)$ a branch of the Riemann surface (3.56). In this case, equations (4.33) and the last equation in (4.32) determine the derivatives $\partial F_0/\partial t_m$ in terms of the coefficients of V and f, g; these are Whitham-type conditions analogous to those of [46]. As shown above, the derivatives $\frac{\partial F_0}{\partial S_\alpha}$ are also determined by the Riemann surface. Together with the Whitham relations, this allows one to determine the planar generating function $F_0(t,S)$ up to an additive constant,[10] given the planar limits $l^{(0)}(t,S)$ and $\Pi(t,S)$.

5. First order analysis of the loop equations

Consider the first order large-N expansion of the microcanonical generating function $F(t,S)$ of the log-deformed model:

$$F(t,S) = F_0(t,S) + \frac{1}{N}F_1(t,S) + O\left(N^{-2}\right). \tag{5.1}$$

The aim of this section is to show that the contribution F_1 can be obtained by differentiation with respect to the parameter t_{-1} in the potential (3.11):

$$F_1(t,S) = \frac{s}{2}\frac{\partial F_0(t,S)}{\partial t_{-1}} + \Psi(t_{-1}, S_0). \tag{5.2}$$

5.1 The resolvent to order $1/N$

To establish this, we first note that the multi-point correlators of the resolvent ω differ from the corresponding product of one point functions only at order $1/N^2$:

$$\langle\omega(z)^2\rangle = \langle\omega(z)\rangle^2 + O\left(\frac{1}{N^2}\right), \qquad \langle\omega(z)^2\omega(-z)\rangle = \langle\omega(z)\rangle^2\langle\omega(-z)\rangle + O\left(\frac{1}{N^2}\right) \tag{5.3}$$

and so forth. Consider the $1/N$–expansion:

$$\langle\omega(z)\rangle = \omega_0(z) + \frac{1}{N}\omega_1(z) + \mathcal{O}\left(\frac{1}{N^2}\right). \tag{5.4}$$

Inserting this and (5.3) into the integral formulation (3.50) of the loop equations we obtain the planar relations:

$$\begin{aligned}
\oint_\gamma \frac{dx}{2\pi i}\frac{2xV'(x)}{z^2-x^2}\omega_0(x) &= \omega_0(z)^2 + \omega_0(z)\omega_0(-z) + \omega_0(-z)^2\,, \\
\oint_\gamma \frac{dx}{2\pi i}\frac{2xV'(x)}{z^2-x^2}\omega_0(x)\omega_0(-x) &= \omega_0(z)^2\omega_0(-z) + \omega_0(-z)^2\omega_0(z)
\end{aligned} \tag{5.5}$$

and the $O(1/N)$ constraints:

$$\begin{aligned}
&\oint_\gamma \frac{dx}{2\pi i}\frac{2xV'(x)}{z^2-x^2}\omega_1(x) + s\oint_\gamma \frac{dx}{2\pi i}\frac{\omega_0(x)}{z^2-x^2} = \\
&\qquad = [2\omega_0(z) + \omega_0(-z)]\omega_1(z) + +[2\omega_0(-z) + \omega_0(z)]\omega_1(-z)\,, \\
&\oint_\gamma \frac{dx}{2\pi i}\frac{2xV'(x)}{z^2-x^2}[\omega_0(-x)\omega_1(x) + \omega_0(x)\omega_1(-x)] + s\oint_\gamma \frac{dx}{2\pi i}\frac{\omega_0(x)\omega_0(-x)}{z^2-x^2} = \\
&\qquad = [2\omega_0(z)\omega_0(-z) + \omega_0(-z)^2]\omega_1(z) + [2\omega_0(-z)\omega_0(z) + \omega_0(z)^2]\omega_1(-z)\,.
\end{aligned} \tag{5.6}$$

[10] We stress that equations (3.26) only determine $F_0(t,S)$ up to the addition of an arbitrary function of t_m. One needs the Whitham constraints in order to fix the t-dependence of F_0.

On the other hand, the filling fraction conditions give:

$$\oint_{\gamma_i} \frac{dz}{2\pi i}\omega_0 = S_i \quad (i = 0\ldots d)\,, \qquad \oint_{\tilde{\gamma}_j} \frac{dz}{2\pi i}\omega_0 = \frac{1}{2}\tilde{S}_j \quad (j = 1\ldots\delta)\,, \qquad \oint_{\tilde{\gamma}_0^+} \frac{dz}{2\pi i}\omega_0 = \frac{1}{2}\tilde{S}_0\,, \tag{5.7}$$

and

$$\oint_{\gamma_i} \frac{dz}{2\pi i}\,\omega_1 = 0 \quad (i = 0\ldots d)\,, \qquad \oint_{\tilde{\gamma}_j} \frac{dz}{2\pi i}\,\omega_1 = 0 \quad (j = 1\ldots\delta)\,, \qquad \oint_{\tilde{\gamma}_0^+} \frac{dz}{2\pi i}\,\omega_1 = 0\,. \tag{5.8}$$

These include the constraints for the log-deformed model arising for the cuts shown in figure 5.

Suppose that $\omega_0(z)$ is a solution to (5.5) and (5.7) for the potential (3.11). By differentiating the planar loop equations (5.5) as well as (5.7) with respect to t_{-1} one can check that $\omega_1^p := \frac{s}{2}\frac{\partial\omega_0}{\partial t_{-1}}$ is a particular solution of both (5.6) and (5.8). To show that this is the unique solution, it suffices to check that the homogeneous system obtained from (5.6) by dropping the terms $s\oint_\gamma \frac{dx}{2\pi i}\frac{\omega_0(x)}{z^2-x^2}$ and $s\oint_\gamma \frac{dx}{2\pi i}\frac{\omega_0(x)\omega_0(-x)}{z^2-x^2}$ from the two left hand sides admits only the trivial solution when supplemented by the constraints (5.8). To establish this, notice that the homogeneous system associated with (5.6) results from the planar equations (5.5) if one performs infinitesimal variations of $\omega_1 = \delta\omega_0$, such that the variations are independent of the coefficients $t_{-1}\ldots t_d$ of the potential (3.11). Such variations of ω_0 arise by changing the filling fraction parameters S_α, which give a parameterization of that part of the moduli space of our Riemann surface which results by varying the quantities f, g of subsection 3.3. Therefore, the solution space to the homogeneous system associated with (5.6) is the tangent space $\mathcal{T}$ to this part of the moduli space of our surface. Since the corresponding moduli space is parameterized by S_α, the dimension of this solution space equals $d+\delta+2$. On the other hand, relations (5.8) give $d+\delta+2$ linearly independent equations for such solutions. These equations select the vanishing solution of the homogeneous system associated with (5.6). Therefore, the homogeneous system obtained from (5.6) has no nontrivial solutions when supplemented by the constraints (5.8). This shows that the solution of (5.6) and (5.8) is uniquely determined and equals ω_1^p.

We conclude that the solution of the loop equations has the form:

$$\langle\omega(z)\rangle = \omega_0(z) + \frac{1}{N}\omega_1(z) + \mathcal{O}\left(\frac{1}{N^2}\right)\,, \tag{5.9}$$

where:

$$\omega_1(z) = \frac{s}{2}\frac{\partial\omega_0(z)}{\partial t_{-1}}\,. \tag{5.10}$$

5.2 The $1/N$ correction to the microcanonical generating function

Next we show that a similar relation holds for the $1/N$-expansion of F. To simplify notation we label all filling fractions S, $\tilde{S}$ by S, keeping the convention that S_0 denotes the filling fraction of I_0 in figure 5. Equation (5.10) was derived under the assumption that the nonsingular part W of our potential contains a finite number of terms d. Since (5.10) must hold for any value of d, it is clear that it also holds if we formally allow W to contain an infinity of terms. Therefore, let us now take $W(z) = \sum_{m=0}^{\infty} \frac{t_m}{m+1} z^{m+1}$. Defining the loop insertion operator $\frac{d}{dW(z)} = \sum_{m=0}^{\infty} \frac{m+1}{z^{m+1}} \frac{\partial}{\partial t_m}$ (see e.g. [50]), we then have $(m+1)\frac{\partial F}{\partial t_m} = \frac{1}{N}\langle tr M^{m+1}\rangle$, for all $m \geq 0$, which gives the standard relation:

$$\langle \omega(z)\rangle = -\frac{d}{dW(z)} F \tag{5.11}$$

upon expanding $\omega(z) = \frac{1}{N}\langle \mathrm{tr}\,\frac{1}{z-M}\rangle = \frac{1}{N}\sum_{m=0}^{\infty} \frac{1}{z^{m+1}} \frac{\langle \mathrm{tr}\, M^{m+1}\rangle}{m+1}$. Here F is the microcanonical generating function. Expanding $F = \sum_{j\geq 0} \frac{1}{N^j} F_j$, this gives $\omega_j = -\frac{d}{dW(z)} F_j$. Therefore, relation (5.10) implies:

$$F_1(t_{-1}, \dots t_\infty, S_0 \dots S_\infty) = \frac{s}{2}\frac{\partial F_0(t_{-1} \dots t_\infty, S_0 \dots S_\infty)}{\partial t_{-1}} + \Psi(t_{-1}, S_0 \dots S_\infty)\,, \tag{5.12}$$

where Ψ is a *universal* function of $t_{-1}, S_0 \dots S_\infty$ which does not depend on $(t_j)_{j\geq 0}$. Since this function is completely independent of $t_0 \dots t_\infty$, it knows nothing about which of these coefficients are zero. To determine it, let us consider a potential (3.11) with $W = 0$; this amounts to setting $t_j = 0$ for all $j \geq 0$. In this case, the associated Riemann surface (3.85) has a single independent period, namely S_0. It follows that the function Ψ does not depend on the variables $S_1 \dots S_\infty$. Thus one has $\Psi = \Psi(t_{-1}, S_0)$. Returning to equation (5.12), we now take W to be of the form (3.11) and obtain:

$$F_1(t_{-1}, t_0 \dots t_d, S_0 \dots S_{d+\delta+1}) = \frac{s}{2}\frac{\partial F_0(t_{-1}, t_0 \dots t_d, S_0 \dots S_{d+\delta+1})}{\partial t_{-1}} + \Psi(t_{-1}, S_0)\,. \tag{5.13}$$

Since we are ultimately interested in the case $t_{-1} = 0$, let us set $S_0 = S_0(t_{-1})$ with $S_0(0) = 0$ in equation (5.13). Then the limit $t_{-1} \to 0$ gives:

$$F_1(0, t_0 \dots t_d, 0, S_1 \dots S_{d+\delta+1}) = \frac{s}{2}\left[\frac{\partial F_0(t_{-1}, t_0 \dots t_d, S_0 \dots S_{d+\delta+1})}{\partial t_{-1}}\right]\Big|_{t_{-1}=0, S_0=0} + \Psi(0,0)\,. \tag{5.14}$$

Therefore, the microcanonical generating function of the model with $t_{-1} = 0$ satisfies:

$$F_1 = \frac{s}{2}\frac{\partial F_0}{\partial t_{-1}}\Big|_{t_{-1}=0} + \mathrm{constant}\,. \tag{5.15}$$

This relation is reminiscent (though somewhat different in character) of similar equations satisfied by the (microcanonical) generating function of SO(N) and Sp($N/2$) one-matrix models [22, 28].

6. Comparison with the field theory description

In this section we want to compute the leading terms in the effective superpotential from our matrix model. The strategy to do so will be the following. We will take the superpotential of the gauge theory in which all the fields are complex. So to start with, there is no hermiticity constraint on M and also Q and $\bar{Q}$ are unrelated. The gauge symmetry will be given by the complexified gauge group just as in the discussion of the classical moduli space of the gauge theory. After imposing an appropriate gauge fixing à la BRST we then choose a suitable real section in the configuration space and perform the path integral along this real section. This is completely analogous to the situation in [7, 18] where in a two-cut vacuum of the one-matrix model one was forced to expand around hermitean and anti-hermitean matrices respectively. This somewhat ad hoc procedure has been justified recently in [20].

According to the DV conjecture, the leading logarithmic terms in the superpotential can be computed from the "non-perturbative" part of the matrix-model partition function. Actually the notion of non-perturbative should be understood here in the sense that we compute the one-loop approximation to the partition function, taking all couplings of terms involving the product of more than two matrices to zero. In other words, we will expand the matrix model around the background only up to terms that are bilinear in the fluctuating fields. In order to keep things simple but also capture the essential physics we choose to expand around the vacuum with vevs given by

$$\langle M\rangle = \mathrm{diag}(\mathbf{0}_{N_0}, a\mathbf{1}_{N_1}, b\mathbf{1}_{N_2}, -b\mathbf{1}_{N_2})\,; \tag{6.1}$$

and

$$\langle Q\rangle = \begin{pmatrix} \mathbf{E}_s & & & \\ & \mathbf{0} & & \\ & & \mathbf{0} & \mathbf{1}_{N_2} \\ & & s\mathbf{1}_{N_2} & \mathbf{0} \end{pmatrix}; \qquad \langle \bar{Q}\rangle = \begin{pmatrix} -sW'(0)\mathbf{E}_s & & & \\ & \mathbf{0} & & \\ & & \mathbf{0} & -sW'(b)\mathbf{1}_{N_2} \\ & & -W'(b)\mathbf{1}_{N_2} & \mathbf{0} \end{pmatrix} \tag{6.2}$$

The $N_0 \times N_0$ matrix $\mathbf{E}_s$ is defined as in section 2. The constants a and b are solutions to the equations $W'(a) = 0$ and $W'(b) = W'(-b)$. This vacuum is the simplest generic one in the sense that each of the different types of vacua we found in the analysis of the classical moduli space appears precisely once.

6.1 Fixing the gauge à la BRST

The gauge transformation are

$$\delta M = i[\Lambda, M]\,, \qquad \delta Q = i\left(\Lambda Q + Q\Lambda^T\right) \qquad \delta\bar{Q} = -i\left(\Lambda^T\bar{Q} + \bar{Q}\Lambda\right). \tag{6.3}$$

Since we are interested only in the quadratic part of the action we can also linearize the gauge transformations around the background. Furthermore from now on we will decompose all matrices in blocks of sizes $N_i \times N_j$ with $i, j \in 0, \ldots, 3$ and denote the fluctuations of the matrices around the vevs with lower case letter, i.e. $M = \langle M\rangle + m$, $Q = \langle Q\rangle + q$ and $\bar{Q} = \langle\bar{Q}\rangle + \bar{q}$. The linearized gauge transformations then take the form

$$\hat{\delta} m_{ij} = i\big(\langle M\rangle_{jj} - \langle M\rangle_{ii}\big)\Lambda_{ij} \tag{6.4}$$

$$\hat{\delta} q_{ij} = i\left(\Lambda_{ik}\langle Q\rangle_{kj} + \langle Q\rangle_{ik}(\Lambda_{jk})^T\right) \tag{6.5}$$

$$\hat{\delta}\bar{q}_{ij} = -i\left((\Lambda_{ki})^T\langle\bar{Q}\rangle_{kj} + \langle\bar{Q}\rangle_{ik}\Lambda_{kj}\right). \tag{6.6}$$

From these transformation we see that we can choose the gauge

$$\begin{aligned} m_{ij} &= 0\,; \quad \text{for} \quad j \neq i\,, \\ q_{23} &= q_{00} = 0\,. \end{aligned} \tag{6.7}$$

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The gauge fixing is now implemented in standard fashion by promoting the gauge transformations to BRST transformations, i.e. replacing the gauge parameters by ghosts. We also need to introduce antighosts $\bar{C}$ and Lagrange multiplier fields B. The gauge fixing lagrangian is given by

$$S_{gf} = \mathbf{s}\ \mathrm{Tr}\left(\sum_{i\neq j}(\bar{C}_{ij}m_{ij}) + \bar{C}_q q_{23} + \bar{C}_{00}q_{00}\right). \tag{6.8}$$

The BRST transformations for the antighosts and Lagrange multipliers are

$$\begin{aligned} \mathbf{s}\,\bar{C}_{ij} &= B_{ij}\,, \qquad \mathbf{s}\,B_{ij} = 0\,, \\ \mathbf{s}\,\bar{C}_q &= B_q\,, \qquad \mathbf{s}\,B_q = 0\,, \\ \mathbf{s}\,\bar{C}_{00} &= B_{00}\,, \qquad \mathbf{s}\,B_{00} = 0\,. \end{aligned} \tag{6.9}$$

Explicitly the gauge fixing action is

$$\begin{aligned} S_{gf} = \mathrm{Tr}\Big[&\sum_{i\neq j} B_{ij}m_{ij} + i(\langle M\rangle_{ii} - \langle M\rangle_{jj})\bar{C}_{ij}C_{ij} + B_q q_{23} - i\bar{C}_q(C_{22}+C_{33}^T) + \\ &+ B_{00}q_{00} - i\bar{C}_{00}(C_{00}\mathbf{E}_s + \mathbf{E}_s C_{00})\Big]\,. \end{aligned} \tag{6.10}$$

The the residual gauge group is

$$G_s = \mathrm{U}(N_1)\times \mathrm{U}(N_2)\times \begin{cases} \mathrm{SO}(N_0) & s=+1 \\ \mathrm{Sp}(N_0/2) & s=-1 \end{cases}\,. \tag{6.11}$$

Notice that the $\mathrm{U}(N_2)$ group is diagonally embedded in $\mathrm{U}(N_2)\times\mathrm{U}(N_2)$ such that the rank of the original $\mathrm{U}(N)$ gauge group is $N = N_1 + 2N_2 + N_0$. According to this structure of the gauge group only the combination $C_{22}+C_{33}^T$ appears in the gauge fixed action. Similarly only the ghosts components of C_{00} that do not correspond to SO/Sp residual gauge transformations propagate.

6.2 Gaussian approximation of partition function and microcanonical generating function

Expanding the action around the background up to terms that are bilinear in the fields gives

$$\begin{aligned} S^{(2)} = N\Bigg(&N_1W(a) + N_2(W(b)+W(-b)) + N_0W(0) + \frac{1}{2}\sum_{i=0}^{3}W''(\langle M\rangle_{ii})\,\mathrm{Tr}(m_{ii}^2)\Bigg) + \\ &+\sum_{i>j}(\langle M\rangle_{ii}+\langle M\rangle_{jj})\,\mathrm{Tr}(\bar{q}_{ij}q_{ji}) + \sum_{i=0}^{3}\langle M\rangle_{ii}\,\mathrm{Tr}(\bar{q}_{ii}q_{ii}) + \\ &+\mathrm{Tr}\left(\bar{q}_{32}\left(m_{22}+m_{33}^T\right)\right) + \frac{1}{2}\mathrm{Tr}\left(\bar{q}_{00}\left(m_{00}\mathbf{E}_s + \mathbf{E}_s m_{00}^T\right)\right)\,. \end{aligned} \tag{6.12}$$

In this expression we used already that $q_{23}=0$, $q_{00}=0$ and $m_{ij}=0$ for $i\neq j$. Notice that this gauge fixing is also important for another reason. Even without our gauge choice

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the blocks q_{23}, q_{00} do not contribute to $S^{(2)}$. This is because $\langle M\rangle_{22} = b = -\langle M\rangle_{22}$ and $\langle M\rangle_{00} = 0$. Fixing the gauge in the way we did eliminates this problem. Moreover, the fields $\bar{q}_{23}$ and $\bar{q}_{00}$ act now as Lagrange multipliers implementing the constraints

$$m_{22} = -m_{33}^T \qquad \text{and} \qquad m_{00}\mathbf{E}_s + \mathbf{E}_s m_{00}^T = 0\,. \tag{6.13}$$

This restricts the fluctuations of M precisely in such a way as to account for a field in the adjoint representation of the diagonally embedded $\mathrm{U}(N_2)$ gauge group and another field in the adjoint representation of $\mathrm{SO}(N_0)$ or $\mathrm{Sp}(\frac{N_0}{2})$ respectively.

In the definition of the matrix model in section 2 we had to implement also a reality constraint of the form $M - M^\dagger = 2i\varepsilon\mathbf{1}$ and $\bar{Q} = -iQ^\dagger$. However, for the present calculation it is more convenient to choose different sections to ensure that all exponentials are decaying. For instance if a sits at a maximum of $W(x)$ we will choose m_{11} anti-hermitean. Similarly, we choose $\bar{q}_{ij} = q_{ij}^\dagger$ or $\bar{q}_{ij} = -q_{ij}^\dagger$ depending on the sign of the coefficient of $\bar{q}_{ij}q_{ij}$ in $S^{(2)}$. For the ghosts we choose either $\mathrm{Re}(C_{ij}) = \mathrm{Re}(\bar{C}_{ij}) = 0$ or $\mathrm{Im}(C_{ij}) = \mathrm{Im}(\bar{C}_{ij}) = 0$.

In addition we can not relate all blocks of q to blocks of $\bar{q}$ by a conjugation. Demanding for example $\bar{q}_{32} = q_{23}^\dagger$ would automatically set $\bar{q}_{32}$ to zero because of the gauge fixing. We do need these fields however in order to implement the constraints (6.13) on the fluctuations of M. Therefore we will demand $\bar{q}_{32}$ imaginary and $\bar{q}_{00}$ imaginary symmetric or antisymmetric depending on s. After integrating out the Lagrange multipliers B_{ij}, B_q, B_{00}, $\bar{q}_{32}$ and $\bar{q}_{00}$ the partition function of the gauge fixed matrix model is given by

$$Z = \frac{1}{\mathrm{vol}(G_s)} \int \prod_{i\in\{0,1,2\}} dm_{ii} \prod_{ij\neq\left\{\substack{(23)\\(00)}\right.} dq_{ij}\; dC\; d\bar{C}\; e^{-(S^{(2)}+S_{gf})} \tag{6.14}$$

Performing the gaussian integrations gives

$$Z = \frac{1}{\mathrm{vol}(G_s)} Z_M\, Z_{\text{ghosts}}\, Z_{\text{matter}}\,, \tag{6.15}$$

where

$$\begin{aligned} Z_M &= \left(\frac{2\pi}{NW''(a)}\right)^{\frac{N_1^2}{2}} \left(\frac{2\pi}{N(W''(b)+W''(-b))}\right)^{\frac{N_2^2}{2}} \left(\frac{2\pi}{NW''(0)}\right)^{\frac{N_0}{4}(N_0-s)}\,, \\ Z_{\text{ghosts}} &= \left(a^2-b^2\right)^{2N_1N_2} a^{2N_1N_0} (2b)^{2N_2^2} b^{4N_2N_0}\,, \\ Z_{\text{matter}} &= \left(\frac{\pi}{a^2-b^2}\right)^{N_1N_2} \left(\frac{\pi}{a}\right)^{N_1N_0+\frac{N_1}{2}(N_1+s)} \left(\frac{\pi}{b}\right)^{2N_2N_0+N_2(N_2+s)}\,. \end{aligned} \tag{6.16}$$

The microcanonical generating function is defined through $N^2F = -\log(Z)$. In order to compute it we need the logarithms of the volume of the residual gauge group. These are given by

$$\begin{aligned} \log(\mathrm{vol}(\mathrm{U}(N))) &= -\frac{N^2}{2}\left(\log(N) - \frac{3}{2} - \log(2\pi)\right) + O(N^0)\,, \\ \log(\mathrm{vol}(H_s(N))) &= -\frac{N^2}{4}\left(\log(N) - \frac{3}{2} - \log(2\pi)\right) + \\ & \qquad s\frac{N}{4}\left(\log(N) - 1 + \log(4) - \log(2\pi)\right) + O(N^0)\,, \end{aligned} \tag{6.17}$$

where H_s is SO(N) for $s = 1$ and Sp$(\frac{N}{2})$ for $s = -1$. According the the DV conjecture we next introduce the filling fractions $S_i = N_i/N$ and define the masses $m(a) = W''(a)$, $m(b) = (W''(b) + W''(-b))$ and $m(0) = W''(0)$. The $1/N$ expansion of the microcanonical generating function we denote by

$$F = \left(F_0 + \frac{1}{N}F_1 + O\left(\frac{1}{N^2}\right)\right). \tag{6.18}$$

The complete microcanonical generating function is

$$F = F_M + F_{\text{ghosts}} + F_{\text{matter}} \tag{6.19}$$

where

$$\begin{aligned} F_M &= -\frac{S_1^2}{2}\log\left(\frac{S_1}{e^{\frac{3}{2}}m(a)}\right) - \frac{S_2^2}{2}\log\left(\frac{S_2}{e^{\frac{3}{2}}m(b)}\right) - \frac{S_0^2}{4}\log\left(\frac{S_0}{e^{\frac{3}{2}}m(0)}\right) + \\ &\quad + \frac{s}{N}\frac{S_0}{4}\log\left(\frac{4S_0}{em(0)}\right), \\ F_{\text{matter}} &= -S_1S_2\log\left(\frac{\pi^2}{a^2-b^2}\right) - \left(S_1S_0 + \frac{S_1^2}{2} - \frac{s}{2N}S_1\right)\log\left(\frac{\pi}{a}\right) - \\ &\quad - \left(2S_2S_0 + S_2^2 + \frac{s}{N}S_2\right)\log\left(\frac{\pi}{b}\right), \\ F_{\text{ghosts}} &= -2S_1S_2\log\left(a^2-b^2\right) - 2S_1S_0\log(a) - 2S_2^2\log(2b) - 4S_2S_0\log(b)\,. \end{aligned} \tag{6.20}$$

6.3 The superpotential in the gaussian approximation

The effective superpotential of the gauge theory can be computed from these expressions as

$$W_{\text{eff}} = \sum_{i\in\{1,2,0\}} N_i\frac{\partial F_0}{\partial S_i} + 4F_1 + \alpha_i S_i\,, \tag{6.21}$$

The subleading $1/N$ term in the microcanonical generating function stems form graphs with topology $\mathbb{RP}^2$. These are present in our theory because of the symmetric/antisymmetric Q matrices. It has been argued in [21, 22, 28] that the $\mathbb{RP}^2$ contribution to F enters the superpotential with a an additional factor 4. This argument has originally been derived in the context of theories with orthogonal and symplectic gauge groups. However, it really only depends on the topology of the Feynman graphs and therefore should go over to our case without essential changes. In any case we will see shortly that (6.21) gives the correct Veneziano-Yankielowicz form of the superpotential. The final ingredient is α. It is given by the coupling of the original U(N) gauge theory

$$\alpha = (N - 2s)\log\left(\frac{\Lambda_{\text{high}}}{\mu}\right). \tag{6.22}$$

The factor $N - 2s$ is the one-loop beta function coefficient of the original gauge theory. The scale μ is the scale where the potential $W(\phi)$ becomes relevant. For notational simplicity we will set this scale to one. Alternatively we could have introduced a scale in the matrix integral such that the S_i obtain the correct dimension three as is appropriate for the interpretation as gaugino condensate. The α_i are then given by $\alpha_{0,1} = \alpha$ and $\alpha_2 = 2\alpha$, where the last factor two stems from the diagonal embedding of the gauge group $\mathrm{U}(N_2)$. Collecting now all the terms we find the effective low energy superpotential

$$\begin{aligned} W_{\text{eff}} = {} & S_1 \log\left(\frac{\Lambda_{\text{low},1}^{3N_1}}{S_1^{N_1}}\right) + S_2 \log\left(\frac{\Lambda_{\text{low},2}^{3N_2}}{S_2^{N_2}}\right) + \frac{S_0}{2} \log\left(\frac{\Lambda_{\text{low},0}^{3(N_0-2s)}}{S_0^{N_0-2s}}\right) + \\ & + S_1 \log\left(\frac{e^{N_1}}{\pi^{N+2s}}\right) + S_2 \log\left(\frac{e^{N_2}}{\pi^{2(N-2s)-2N_2}}\right) + \frac{S_0}{2} \log\left(\frac{4^{2s} e^{N_0-2s}}{\pi^{2N_1+4N_2}}\right). \end{aligned} \tag{6.23}$$

The relation between the low- and high energy scales is

$$\begin{aligned} \Lambda_{\text{low},1}^{3N_1} &= m(a)^{N_1}\, a^{N_1+2s} \left(a^2 - b^2\right)^{-N_2} a^{-N_0}\, \Lambda_{\text{high}}^{N-2s}, \\ \Lambda_{\text{low},2}^{3N_2} &= m(b)^{N_2} \left(a^2 - b^2\right)^{-N_1} b^{2N_2+4s-2N_0}\, (2b)^{-4N_2}\, \Lambda_{\text{high}}^{2(N-2s)}, \\ \Lambda_{\text{low},0}^{\frac{3}{2}(N_0-2s)} &= m(0)^{\frac{N_0}{2}-s}\, a^{-N_1}\, b^{-2N_2}\, \Lambda_{\text{high}}^{N-2s}. \end{aligned} \tag{6.24}$$

Thus the effective superpotential coincides with the one obtained in section 2 up to the linear terms in the S_i. Of course these linear terms can not be determined by naive threshold matching.

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A. Comparison with the $O(n)$ and A_2 matrix models

A.1 Comparison with the $O(n)$ model

Let us compare our models with symmetric and antisymmetric matter with the $O(n)$ model analyzed in [42, 43]. The difference is the (M, Q) interaction term, which reads in the $O(n)$ model

$$\sum_{i=1}^{n} \operatorname{tr} Q_i^\dagger M Q_i\,, \tag{A.1}$$

where $Q_i^\dagger = Q_i$ are hermitean. After diagonalizing M in the relevant case for comparison[11] $n = 1$, the gaussian integration over the real Q_{ii} leads in our conventions to $s = 0$. To summarize one has $s = 1$ for Q_{ii} complex, $s = 0$ for Q_{ii} real and $s = -1$ for $Q_{ii} = 0$ and this affects only the definition of $\mathrm{U}(x)$ (3.12) while the rest of the derivation of the loop equations remains the same. The quadratic loop equation (3.33) agrees in the large-N-limit with the one in [42, 43]. A cubic loop equation does not appear in [42, 43] explicitly. The second order loop equations generalize very directly to general n. The third order

[11] For a special cubic $V(x)$ this corresponds to Ising model on a random lattice [42].

loop equation should become an $n+1$ order loop equation. From the gauge theory point of view the $n=1$ model corresponds to adding matter in the adjoint representation, a spectrum which emerges e.g. from an $\mathcal{N}=4$ theory. Adding more matter $n>1$ leads to non asymptotically free cases.

A.2 Comparison with the A_2 quiver theory

Let us next compare the situation to the ADE quiver theories, in particular to A_2, which is the double cover of the models discussed so far. The ADE matrix models where discussed in [38, 39, 41] and further considered in the context of the DV conjecture in [2, 51, 52]. The action of the ADE quiver matrix models is:

$$S_{\mathrm{ADE}} = \sum_{i=1}^{r} \mathrm{tr}\, W_{(i)}\left(M^{(i)}\right) + \sum_{i,j=1}^{r} s_{ij}\,\mathrm{tr}\, Q_{(ij)} M^{(j)} Q_{(ji)}\,. \tag{A.2}$$

Here one has an $N^{(i)} \times N^{(i)}$ matrix $M^{(i)}$ for every node in the Dynkin diagram of the ADE Lie algebra and the $Q_{(ij)}$ are $N^{(i)} \times N^{(j)}$ matrices transforming in the $(N^{(i)}, \bar{N}^{(j)})$ representation and fulfilling $Q^{\dagger}_{(ij)} = Q_{(ji)}$. For a given ordering of the nodes $s_{ij} = -s_{ji} = 1$ if node i and node j are linked in the diagram and $s_{ij} = 0$ otherwise. After diagonalizing all $M^{(i)}$ the gaussians in the Q-fields appear with different signs. Let us assume $s_{ij} = 1$ and $Q_{ji} = iQ^{\dagger}_{ij}$. Then we have to shift the $M^{(i)}$ eigenvalues into the upper half-plane whereas the $M^{(j)}$ eigenvalues have to be shifted into the lower half-plane ! For $s_{ij} = -1$ we can still shift the $M^{(i)}$ eigenvalues into the upper half-plane but we then have to demand $Q_{ji} = -iQ^{\dagger}_{ij}$. After choosing these real slices through the matrix configuration space the gaussian integrations over all Q-fields can be performed.

$$N^2 S_{\mathrm{ADE}} = N \sum_{i=1}^{r} \sum_{k=1}^{N^{(i)}} W_{(i)}\left(\lambda^{(i)}_k\right) - \sum_{i=1}^{r} \sum_{k\neq l} \ln(\lambda^{(i)}_k - \lambda^{(i)}_l) + \sum_{i<j}^{r} \sum_{k,l} \ln(\lambda^{(i)}_k - \lambda^{(j)}_l)^{|s_{ij}|}\,. \tag{A.3}$$

The orientifold projections are obtained by $M^{(1)} = -\left(M^{(2)}\right)^T$ and $\bar{Q}^T_{12} = sQ_{21}$ and lead to the symmetric or antisymmetric matter. Identifying $Q_{12} = Q_{21}$ leads to the $n=1$ $O(n)$ model. The action on M identifies for all cases $\lambda^{(1)}_k = -\lambda^{(2)}_k$, while the projection of the Q changes the result of the gaussian integration starting from (A.2). Let us sketch the derivation of the exact loop equations for $S := S_{A_2}$ in the following. The e.o.m. for the $\lambda^{(1)}_k$ eigenvalues read:

$$N\frac{\partial S_{\mathrm{ADE}}}{\partial \lambda^{(1)}_k} = W'_{(1)}\left(\lambda^{(1)}_k\right) - \frac{1}{N}\sum_{l(\neq k)}^{N_1} \frac{2}{\lambda^{(1)}_k - \lambda^{(1)}_l} + \frac{1}{N}\sum_{l}^{N_2} \frac{1}{\lambda^{(1)}_k - \lambda^{(2)}_l}\,, \tag{A.4}$$

with an analogous expression for the $\lambda^{(2)}_k$.

Let us define the resolvents for the two matrices as $\omega_i(z) = \frac{1}{N}\sum_k^{N_i} \frac{1}{z-\lambda^{(i)}_k}$, $i=1,2$ and choose $\Psi^{(i)}_k = \frac{1}{z-\lambda^{(i)}_k}$, $i=1,2$ in (3.30) to obtain two Ward identities, which can be simplified with analogous equations to (3.31). Adding these two Ward identities one gets the quadratic loop equation:

$$\langle \omega_1(z)^2 - \omega_1(z)\omega_2(z) + \omega_2(z)^2 - W'_1(z)\omega_1(z) - W'_2(z)\omega_2(z) + f_1(z) + f_2(z)\rangle = 0\,. \tag{A.5}$$

Here we defined

$$f_i(z) = \frac{1}{N}\sum_{k=1}^{N_i} \frac{W_i'(z) - W_i'(\lambda_k^{(i)})}{z - \lambda_k^{(i)}}\,, \qquad i = 1,2\,. \tag{A.6}$$

The derivation of the cubic loop equations is likewise very similar to the discussion in the orientifolded model. One uses $\Psi_k^{(1)} = \frac{1}{N}\sum_{m=1}^{N_2} \frac{1}{\lambda_k^{(1)} - \lambda_m^{(2)}} \frac{1}{z - \lambda_k^{(1)}}$ in (3.30) and simplifies the Ward identity with (3.41). Subtracting the same equation, but with indices $(1) \leftrightarrow (2)$ exchanged, yields the cubic loop equation:

$$\begin{aligned}\langle \omega_1(z)\omega_2(z)^2 - \omega_1(z)^2\omega_2(z) + W_1'(z)(\omega_1(z)^2 + f_1(z) - W_1'(z)\omega_1(z)) - \\ -W_2'(z)(\omega_2(z)^2 + f_2(z) - W_2'(z)\omega_2(z)) + g_2(z) - g_1(z)\rangle = 0\,,\end{aligned} \tag{A.7}$$

where

$$g_i(z) = \frac{1}{N^2}\sum_{k=1}^{N_1}\sum_{m=1}^{N_2} \frac{W_i'(z) - W_i'(\lambda_k^{(i)})}{(\lambda_k^{(i)} - \lambda_m^{(j\neq i)})(z - \lambda_k^{(i)})}\,, \tag{A.8}$$

Note that the cubic loop equation for the quiver model does not depend explicitly on $1/N$. In contrast the cubic loop equation for the $O(1)$ model does contain a subleading $1/N^2$ term and the orientifold model contains even terms at order $1/N$. This is consistent with the fact that $1/N$ terms come from non-orientable diagrams and these are absent of course in the A_2 and $O(1)$ model. According to the DV conjecture, terms of order $1/N^2$ are related to gravitational corrections. The effective pure field theoretical superpotential of the orbifold model can therefore be obtained by taking the effective superpotential of the A_2 model and simply identifying the relevant variables, e.g. $S_i^{(1)} = S_i^{(2)}$. However, in the terms related to the gravitational couplings this procedure would not give a correct result, since then one has to take into account the additional $1/N^2$ terms in the cubic loop equation! It would be interesting to investigate this issue further and to see if these extra terms can be related to the presence of twisted sectors in the dual string model.

A.2.1 The large-N Riemann surface for the A_2-quiver

Here we will take the large-N limit of the loop equations for the A_2 quiver. We will see that the Riemann surface governing the A_2 model is not hyperelliptic but rather is a triple cover of the z-plane, one of whose branches gives $\omega_{i,0}(z)$. The occurrence of Riemann surfaces with a more complicated sheet structure for multi matrix models is not unexpected [40, 2]. Our derivation from the exact loop equations will provide a proof for the curves of A_2 quiver and the orbifolds thereof.

The leading terms will be extracted from the loop equations (A.5), (A.7). Using the variables[12]

$$\begin{aligned}\omega_1(z) &= u_1(z) - t_1(z)\,, \qquad t_1(z) = -\frac{1}{3}\left(2W_1'(z) + W_2'(z)\right) \\ \omega_2(z) &= -u_2(z) + t_2(z)\,, \qquad t_2(z) = \frac{1}{3}\left(W_1'(z) + 2W_2'(z)\right)\end{aligned} \tag{A.9}$$

[12]In order to keep the notation simple, we use here ω_i for $\omega_{i,0}$.

one obtains the from the large-N-loop equations the affine complex algebraic curve

$$\begin{aligned} u_1(z)^2 + u_1(z)u_2(z) + u_2(z)^2 &= p(z) \\ u_1(z)u_2(z)^2 + u_1(z)^2u_2(z) &= -q(z) \end{aligned} \tag{A.10}$$

where

$$\begin{aligned} p(z) &= t_1^2 + t_1t_2 + t_2^2 - f_1 - f_2 \\ q(z) &= -t_1^2t_2 - t_1t_2^2 + f_1t_2 + f_2t_1 - g_1 + g_2\,. \end{aligned} \tag{A.11}$$

Now (A.10) are the Viete relations for the cubic

$$\begin{aligned} \prod_{i=0}^{2}(u(z) - u_i) &= u^3 - p(z)u - q(z) \\ &= \prod_{i=0}^{2}(u - t_i(z)) + u(f_1 + f_2) + g_1 - g_2 - f_2t_1 - f_1t_2 = 0\,, \end{aligned} \tag{A.12}$$

where $u_0 = -u_1 - u_2$ and $t_0 = -t_1 - t_2$. As we have shown that f_i and g_i are polynomials in z, (A.12) is an algebraic curve, and can be viewed as a triple cover of the z plane [2]. The perturbations encoded in f_i and g_i can be explicitly related to the eigenvalue densities of the two matrix model. If $d_1 + 1, d_2 + 1$ are the degrees of the potentials W_1, W_2, the degrees of f_i, g_i are $d_i - 1$ respectively. The perturbation parameters introduced by g_i are $\#(g) = \max(d_1, d_2)$ and $\#(f) = d_1 + d_1$ for the f_i. In particular for $d_1 = d_2 = d$ relevant for the modding below we get $3d$, which is is compatible with number of branches calculated in [36]: for equal degree $d + 1$ of the W_i it is dR_+, where R_+ are the number of positive roots for the quiver ADE group. It has also been checked [36] that this number coincides with the number of independent S^3 in $H^3(M)$, where M is the local Calabi-Yau threefold after the large-N transition.

Using the $\mathbb{Z}_2$ identification $\lambda^{(1)} = -\lambda^{(2)}$ on gets

$$\begin{aligned} \omega_1(z) &= \omega(z)\,, \qquad \omega_2(z) = -\omega(-z)\,, \\ W_1'(z) &= W'(z)\,, \qquad W_2'(z) = -W'(-z) \end{aligned} \tag{A.13}$$

etc. Since it was shown in section 3.1 that all log dependence can be absorbed into $V(z)$ we can in general replace $W'(z)$ by $V'(z)$ and recover the large-N limit of the two loop equations with the $\mathbb{Z}_2$ orientifold identification (3.51).

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